

# COHOMOGENEITY-THREE HYPERKÄHLER METRICS ON NILPOTENT ORBITS

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**ABSTRACT.** Let  $\mathcal{O}$  be a nilpotent orbit in  $\mathfrak{g}^{\mathbb{C}}$  where  $G$  is a compact, simple group and  $\mathfrak{g} = \text{Lie}(G)$ . It is known that  $\mathcal{O}$  carries a unique  $G$ -invariant hyperKähler metric admitting a hyperKähler potential compatible with the Kirillov-Kostant-Souriau symplectic form. In this work, the hyperKähler potential is explicitly calculated when  $\mathcal{O}$  is of cohomogeneity three under the action of  $G$ . It is found that such a structure lies on a one-parameter family of hyperKähler metrics with  $G$ -invariant Kähler potentials if and only if  $\mathfrak{g}$  is  $\mathfrak{sp}(3)$ ,  $\mathfrak{su}(6)$ ,  $\mathfrak{so}(7)$ ,  $\mathfrak{so}(12)$  or  $\mathfrak{e}_7$  and otherwise is the unique  $G$ -invariant hyperKähler metric with  $G$ -invariant Kähler potential.

## 1. INTRODUCTION

Let  $G$  be a compact connected semisimple Lie group with Lie algebra  $\mathfrak{g}$  and let  $\mathcal{O}$  be an adjoint nilpotent  $G^{\mathbb{C}}$ -orbit in the complexified Lie algebra  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ . Then  $\mathcal{O}$  carries a canonical  $G^{\mathbb{C}}$ -invariant complex symplectic structure known as the Kirillov-Kostant-Souriau symplectic structure (see eg. [2]). Using an identification of  $\mathcal{O}$  with a set of solutions of Nahm's equations, Kronheimer [17] showed that  $\mathcal{O}$  possesses a  $G$ -invariant hyperKähler metric.

Recall that a Riemannian manifold  $(M, g)$  is *hyperKähler* if it admits three endomorphisms  $I, J$  and  $K$  of the tangent bundle that satisfy the relations of the quaternions (i.e.  $I^2 = J^2 = -1$  and  $IJ = K = -JI$ ), each preserved by the Levi-Civita connection and each turning  $g$  into a Hermitian metric. Necessarily,  $I, J, K$  are integrable so that  $(M, g, I)$  etc. are Kähler structures. As observed by Hitchin [11], replacing the condition  $\nabla I = \nabla J = \nabla K = 0$  with  $d\omega_I = d\omega_J = d\omega_K = 0$ , the almost complex structures  $I, J$  and  $K$  are still integrable; the two-forms  $\omega_I = g(I\cdot, \cdot)$  etc. are called Kähler forms. Note that if one fixes the complex structure  $I$ , then the non-degenerate and closed two-form  $\omega = \omega_J + i\omega_K$  is holomorphic with respect to  $I$ , so  $(M, I)$  carries a

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*complex symplectic structure* defined by  $\omega$ . Taking the  $2n$ 'th power of  $\omega$  one gets a parallel section of the canonical bundle, revealing the well-known fact that hyperKähler metrics are Ricci-flat and Calabi-Yau.

By considering the cotangent bundle  $T^*\mathbb{C}P(n)$ , Calabi [6] gave the earliest non-trivial examples of hyperKähler metrics. Calabi's metric has symmetry group  $U(n+1)$  and the space  $T^*\mathbb{C}P(n)$  is via the complex moment map identified with an adjoint orbit in the semisimple Lie algebra  $\mathfrak{gl}(n+1, \mathbb{C})$ , and it is of cohomogeneity one under the action of  $SU(n+1)$ . Dancer & Swann [8] classified hyperKähler metrics in cohomogeneity one with respect to a compact simple Lie group on manifolds of dimension greater than four. There are two possibilities: the Calabi metric on  $T^*\mathbb{C}P(n)$  and nilpotent orbits of cohomogeneity one in complex simple Lie algebras. A feature of all of these metrics is that they admit an invariant Kähler potential. This aspect makes it natural to consider nilpotent orbits of a given cohomogeneity and with invariant hyperKähler metrics that admit Kähler potentials.

A hyperKähler potential is a function that is simultaneously a Kähler potential of each complex structure compatible with the hyperKähler metric. By Swann [19] it is known that the hyperKähler structure of Kronheimer on a nilpotent orbit admits a  $G$ -invariant hyperKähler potential. In fact, it is the unique  $G$ -invariant hyperKähler potential on  $\mathcal{O}$ , as shown by Brylinski [5].

In this paper we study nilpotent orbits  $\mathcal{O}$  of cohomogeneity three under the action of the compact group  $G$ . We restrict ourselves to the case when  $\mathfrak{g}$  is simple and consider  $G$ -invariant hyperKähler structures on  $\mathcal{O}$  compatible with the Kirillov-Kostant-Souriau symplectic structure, that admit a  $G$ -invariant Kähler potential. Expressing the metric in terms of the Killing form and the Lie bracket we explicitly find the unique  $G$ -invariant hyperKähler potential on each orbit. We show that this potential lies in a larger (one-dimensional) family of  $G$ -invariant hyperKähler metrics with  $G$ -invariant Kähler potential if and only if  $\mathfrak{g}$  is  $\mathfrak{sp}(3)$ ,  $\mathfrak{su}(6)$ ,  $\mathfrak{so}(7)$ ,  $\mathfrak{so}(12)$  or  $\mathfrak{e}_7$ .

For orbits of cohomogeneity two Kobak & Swann [15] found similar results.

It is essential that when  $G$  is not  $SO(7)$  each element of a nilpotent orbit of cohomogeneity three can be moved into a subalgebra of three commuting  $\sigma$ -invariant copies of  $\mathfrak{sl}(2, \mathbb{C})$ , where  $\sigma$  is the compact real structure. This feature is shown using an interesting relation with rank three Hermitian symmetric spaces and it allows one to describe the geometry using representations of  $\mathfrak{sl}(2, \mathbb{C})$ . On the other hand, the cohomogeneity three nilpotent orbit in  $\mathcal{O}_{(3,2^2)} \subset \mathfrak{so}(7, \mathbb{C})$

is special: for generic  $X \in \mathcal{O}_{(3,2^2)}$  the algebra generated by  $X$  and  $\sigma X$  is all of  $\mathfrak{so}(7, \mathbb{C})$ , so a separate direct approach has to be used. We find that there is only a one-dimensional family of hyperKähler metrics with Kähler potential. This is surprising for the following reason. It is known that  $\mathcal{O}_{(3,2^2)} \subset \mathfrak{so}(7, \mathbb{C})$  has a one-to-one correspondence with the cohomogeneity-two nilpotent orbit in  $\mathcal{O}_{(2^4)} \subset \mathfrak{so}(8, \mathbb{C})$  that carries a one-parameter family of  $SO(8)$ -invariant hyperKähler metrics with  $SO(8)$ -invariant Kähler potential. Consequently, reducing the symmetry group from  $SO(8)$  to  $SO(7)$  give us no extra families even though the cohomogeneity changes from two to three.

## 2. GEOMETRY OF NILPOTENT ORBITS

In this section we review some known facts on nilpotent orbits and establish basic notation. We introduce the techniques used to compute cohomogeneities on nilpotent orbits and hopefully clarify the results given in Tables 2 and 3 via examples.

**2.1. Principal Orbits and Cohomogeneities.** If  $G$  is a Lie group acting on some space  $M$  then two orbits  $G \cdot x, G \cdot y \subseteq M$  are said to have the same *orbit type* if the isotropy subgroups  $G_x$  and  $G_y$  are conjugate in  $G$ . If  $M$  is a manifold the following fundamental theorem shows the existence of a maximal orbit type.

**Theorem 2.1** (see [3, 4]). *Let  $M$  be a manifold admitting a differentiable action of a compact Lie group  $G$ . Suppose the orbit space  $M/G$  is connected. Then there exists a maximal orbit type  $(H)$ . That is,  $H$  is conjugate to a subgroup of each isotropy group. The union  $M_{(H)}$  of the orbits of type  $(H)$  is open and dense in  $M$  and  $M_{(H)}/G$  is connected.  $\square$*

An orbit of type  $(H)$ , with  $H$  as in Theorem 2.1, is called a *principal orbit* and the *cohomogeneity* of the action of  $G$  on  $M$  is defined to be the codimension of a principal orbit.

With  $G$  compact each point  $p \in M$  has a slice. This is a submanifold  $S$  such that  $GS \subset M$  is open and there exists a  $G$ -equivariant isomorphism of the  $G$ -spaces  $GS$  and  $G \times_H S$ , where  $H$  is the isotropy group  $G_p$ . Note in particular that if  $M$  is a vector space and  $G$  a compact Lie group then  $GS \cong G \times_H W$  (cf. [4]), where  $W$  is the normal space of the tangent space  $T_p(G \cdot p)$ . This is of great importance to us because we thereby get the equality

$$(2.1) \quad \text{cohom}_G GS = \text{cohom}_H W.$$

**Example 2.2.** Let us use (2.1) to compute the cohomogeneity of some group representations. We start off with a simple example.

(i) Let  $V^n$  denote the  $U(n)$ -module  $\mathbb{C}^n$  via the standard action and let  $\{e_j\}$  be the standard basis of  $V^n$ . The stabilizer in  $U(n)$  of  $e_1$  is

$$H = U(n)_{e_1} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in U(n-1) \right\} \cong U(n-1).$$

As an  $H$ -module we now have  $T_{e_1} V^n = V^n = \mathbb{C} \oplus V^{n-1}$ , and in turn  $\mathfrak{u}(n) \cong \mathbb{R} \oplus V^{n-1} \oplus \mathfrak{u}(n-1)$ , where  $\mathbb{R}, \mathbb{C}$  are trivial modules. Thus,  $T_{e_1}(U(n) \cdot e_1) \cong \mathbb{R} \oplus V^{n-1}$  and we conclude that

$$\text{cohom}_{U(n)} V^n = \text{cohom}_H \mathbb{R} = 1.$$

Since the unitary group acts transitively on the sphere, the above result should come as no surprise. Finding the cohomogeneity of the following representations is less trivial.

(ii) Consider the  $U(n)$  module  $\Lambda^2 V^n = \text{span}_{\mathbb{C}} \{e_i \wedge e_j \mid 1 \leq i < j \leq n\}$ . The stabilizer of  $e_1 \wedge e_2$  is

$$\begin{aligned} H = U(n)_{e_1 \wedge e_2} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in SU(2), B \in U(n-2) \right\} \\ &\cong SU(2) \times U(n-2). \end{aligned}$$

As an  $H$ -module we have  $V^n = S^1 V^2 \oplus V^{n-2}$ , giving us

$$T_{e_1 \wedge e_2} \Lambda^2 V^n = \Lambda^2 V^n = \Lambda^2 (S^1 V^2) \oplus (S^1 V^2 \otimes V^{n-2}) \oplus \Lambda^2 V^{n-2}.$$

Moreover  $\mathfrak{u}(n) = \mathfrak{u}(2) \oplus \mathfrak{u}(n-2) \oplus (S^1 V^2 \otimes V^{n-2})$ , so that

$$T_{e_1 \wedge e_2}(U(n) \cdot (e_1 \wedge e_2)) \cong \mathbb{R} \oplus (S^1 V^2 \otimes V^{n-2}).$$

The cohomogeneity is now

$$\begin{aligned} \text{cohom}_{U(n)} \Lambda^2 V^n &= \text{cohom}_H (\Lambda^2 \mathbb{R}^2 \oplus \Lambda^2 V^{n-2}) \\ &= 1 + \text{cohom}_{U(n-2)} \Lambda^2 V^{n-2} \\ &= \begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd} \end{cases}, \end{aligned}$$

by induction.

(iii) Consider the symmetric product  $S^2 V^3$ . The stabilizer in  $U(3)$  of  $e_1 \vee e_1$  is

$$H = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in U(2) \right\} \cong \mathbb{Z}_2 \times U(2).$$

As  $H$ -modules we have  $V^3 = \mathbb{C} \oplus V^2$ ,  $S^2 V^3 \cong \mathbb{C} \oplus V^2 \oplus S^2 V^2$  and  $\mathfrak{u}(3) = \mathbb{R} \oplus V^2 \oplus \mathfrak{u}(2)$ , so that  $T_{e_1 \vee e_1}(U(3) \cdot e_1 \vee e_1) \cong \mathbb{R} \oplus V^2$ . The cohomogeneity is thus

$$\begin{aligned} \text{cohom}_{U(3)} S^2 V^3 &= \text{cohom}_H (\mathbb{R} \oplus S^2 V^2) \\ &= 1 + \text{cohom}_{U(2)} S^2 V^2 = 3. \end{aligned}$$

**2.2. Adjoint Nilpotent Orbits.** Letting  $\sigma$  denote the conjugation map of  $\mathfrak{g}^{\mathbb{C}}$  with respect to the compact real form  $\mathfrak{g}$ , the following defines a positive definite symmetric bilinear form on the real Lie algebra  $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}}$

$$(X, Y) \mapsto \langle X, \sigma Y \rangle, \quad X, Y \in (\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}},$$

where  $\langle \cdot, \cdot \rangle$  is the negative of the Killing form.

To ease notation we recursively define a multiple bracket  $[[\dots]]$  on the Lie algebra  $(\mathfrak{g}^{\mathbb{C}}, [\cdot, \cdot])$ ,

$$\begin{aligned} [[X_1 X_2]] &= [X_1, X_2], \\ [[X_1 X_2 \dots X_k]] &= [X_1, [[X_2 \dots X_k]]], \quad X_j \in \mathfrak{g}^{\mathbb{C}}. \end{aligned}$$

Via the adjoint action each element  $A \in \mathfrak{g}^{\mathbb{C}}$  generates a vector field  $\xi_A$  on a nilpotent orbit  $\mathcal{O}$ . This is given by  $\xi_A|_X = [[AX]]$ . When there is no danger of confusion, the subscript  $X$  will be suppressed throughout this paper. An important property of these vector fields is the identity  $[[\xi_A \xi_B]] = -\xi_{[[AB]]}$ . As the tangent space of  $\mathcal{O}$  at  $X$  is  $\{[[AX]] | A \in \mathfrak{g}^{\mathbb{C}}\}$ , we see that the complex structure  $I$  is characterized by  $I\xi_A = \xi_{iA}$ .

The nilpotent orbit  $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$  is a complex submanifold with complex structure  $I$  which is inherited from the natural embedding. On  $\mathcal{O}$  the *Kirillov-Kostant-Souriau* symplectic form is given by,

$$\Sigma(\xi_A, \xi_B)_X = \langle [[AB]], X \rangle.$$

We say that a hyperKähler structure  $(g, I, J, K)$  is compatible with  $(\mathcal{O}, \Sigma, I)$  if it satisfies the equation  $\Sigma = \omega_J + i\omega_K$ .

The compact group acts naturally on the non-compact orbit  $\mathcal{O}$  and we shall have a particular interest in the situation with three parameters transverse to the action, that is when  $\mathcal{O}$  is a nilpotent orbit of cohomogeneity three. However, we notice that

*Remark 2.3.* If  $\text{cohom}_G \mathcal{O} = 1$  then  $\mathbb{R}_+ \times G$  acts transitively on  $\mathcal{O}$ .

Orbits in classical Lie algebras are essentially determined by Jordan normal forms of elements in the standard representation, so they are specified by partitions. This characterization is not useful for orbits in exceptional Lie algebras, which are best described by a weighted Dynkin diagram (see [7]).

**2.3. Standard Triples.** A set  $\{H, X, Y\}$  of nonzero elements in  $\mathfrak{g}^{\mathbb{C}}$  is called a *standard triple* if the following bracket relations hold,

$$[[HX]] = 2X, \quad [[HY]] = -2Y \quad \text{and} \quad [[XY]] = H.$$

Since the element  $H$  is a semisimple element of the Lie subalgebra spanned by  $\{H, X, Y\}$ , which in turn is isomorphic to the simple Lie

algebra  $\mathfrak{sl}(2, \mathbb{C})$ , it must be semisimple as an element of  $\mathfrak{g}^{\mathbb{C}}$ . Similarly  $X, Y$  are nilpotent elements of  $\mathfrak{g}^{\mathbb{C}}$ .

The Jacobson-Morozov theorem is of fundamental importance here because it provides access to the representation theory of  $\mathfrak{sl}(2, \mathbb{C})$ ; a detailed proof may be found in [7].

**Theorem** (Jacobson–Morozov). *Let  $\mathfrak{g}^{\mathbb{C}}$  be a semisimple Lie algebra. If  $X$  is a nonzero nilpotent element of  $\mathfrak{g}^{\mathbb{C}}$ , then  $X$  embeds into a standard triple  $\{H, X, Y\}$  of  $\mathfrak{g}^{\mathbb{C}}$ .*  $\square$

Given a compact real form one may obtain a  $\sigma$ -invariant  $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra associated to a nilpotent orbit.

**Proposition 2.4** (Borel). *Let  $\mathcal{O}$  be a nilpotent orbit in  $\mathfrak{g}^{\mathbb{C}}$  and let  $\sigma$  be the conjugation of a compact real form. Then  $\mathcal{O}$  contains an element  $X$  such that  $\{\llbracket X(\sigma X) \rrbracket, X, \sigma X\}$  spans a Lie subalgebra isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$ .*

*Proof.* See [15].  $\square$

**2.4. The Closure Ordering.** There exists a partial order on the set of nilpotent orbits in  $\mathfrak{g}^{\mathbb{C}}$  defined by  $\mathcal{O}' \preceq \mathcal{O}$  if and only if  $\mathcal{O}' \subset \overline{\mathcal{O}}$ . Notice that the boundary of any orbit is the union of orbits of smaller dimensions,

$$(2.2) \quad \overline{\mathcal{O}'} = \bigcup_{\mathcal{O} \preceq \mathcal{O}'} \mathcal{O}.$$

On the level of cohomogeneity it is a pleasure to note the following Theorem by Dancer and Swann [9].

**Theorem 2.5.** *Let  $G$  be a compact simple Lie group. If  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are nilpotent orbits with  $\mathcal{O}_1 \not\preceq \mathcal{O}_2$ , then*

$$\mathrm{cohom}_G \mathcal{O}_1 < \mathrm{cohom}_G \mathcal{O}_2.$$

$\square$

**Example 2.6.** Any element  $X$  of the nilpotent orbit in  $\mathfrak{sl}(n, \mathbb{C})$  characterized by the Jordan normal form  $d = (3, 1^{n-3})$  with  $n \geq 4$  is  $SU(n)$ -conjugated to an element of the form

$$\begin{pmatrix} 0 & s & \lambda & r \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r, s, t \in \mathbb{R}_+ \quad \lambda \in \mathbb{C}.$$

Consequently the cohomogeneity of  $\mathcal{O}_d \subset \mathfrak{sl}(n, \mathbb{C})$  under the action of the compact group  $SU(n)$  is at most 5. Whenever  $n \geq 4$ ,  $\mathcal{O}_d$  is neither minimal nor next-to-minimal (see Section 2.5), so by Theorem 2.5

the cohomogeneity is 3, 4 or 5. Evidently, for  $n = 4$ , the cohomogeneity is no less than 5. But for  $n \geq 5$  we need more information (see Example 2.10).

**2.5. The Minimal Orbit.** When  $\mathfrak{g}^{\mathbb{C}}$  is simple there is a unique nonzero minimal orbit  $\mathcal{O}_{\min} \neq \{0\}$  with respect to the closure ordering (see [7]). In fact  $\mathcal{O}_{\min}$  corresponds to a highest root, so it is the orbit through a nonzero element in a highest root space.

We label the orbits above the minimal orbit via

**Definition 2.7.** (i) The *minimal nilpotent orbit*  $\mathcal{O}_{\min}$  is the unique nilpotent orbit characterized by  $\mathcal{O}_{\min} \preceq \mathcal{O}$  for any nonzero orbit  $\mathcal{O}$ . The minimal orbit will also be referred to as *the height one orbit*.

(ii) The *height* of a nilpotent orbit  $\mathcal{O} \not\preceq \mathcal{O}_{\min}$  is defined by induction. Let  $S_{\mathcal{O}}$  be the set of nilpotent orbits  $\mathcal{O}' \not\preceq \mathcal{O}$  for which there is no nilpotent orbit  $\mathcal{O}''$  satisfying  $\mathcal{O}' \not\preceq \mathcal{O}'' \not\preceq \mathcal{O}$ . Let  $\mathcal{O}_0 \in S_{\mathcal{O}}$  be an orbit of minimal height  $n$ . Then the height of  $\mathcal{O}$  is  $n + 1$ .

A nilpotent orbit of height two will also be referred to as *next-to-minimal*. Notice that the Lie algebra may possess more than one next-to-minimal orbit, as well as orbits of height three.

The minimal orbit is in fact the unique nilpotent orbit of cohomogeneity one under the action of the compact group  $G$ . If  $\alpha^{\#}$  is a highest root we get a *highest root decomposition* by

**Proposition 2.8.** (i) *As an  $\mathfrak{sl}(2, \mathbb{C})_{\alpha^{\#}}$ -module, the Lie algebra  $\mathfrak{g}^{\mathbb{C}}$  decomposes as*

$$(2.3) \quad \mathfrak{g}^{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})_{\alpha^{\#}} \oplus \mathfrak{k}^{\mathbb{C}} \oplus (S^1 \otimes V),$$

where  $\mathfrak{k}^{\mathbb{C}}$  is the centralizer of  $\mathfrak{sl}(2, \mathbb{C})_{\alpha^{\#}}$ ,  $S^1 = S^1 \mathbb{C}^2$  is the fundamental  $\mathfrak{sl}(2, \mathbb{C})_{\alpha^{\#}}$  representation, and  $V$  is a trivial  $\mathfrak{sl}(2, \mathbb{C})_{\alpha^{\#}}$ -module and a non-trivial  $\mathfrak{k}^{\mathbb{C}}$ -module.

(ii) *Under the action of the compact group  $G$  the nilpotent orbit  $\mathcal{O}_{\min}$  is of cohomogeneity one.*

*Proof.* See Kobak and Swann [14]. □

The compact homogeneous space  $W(G) = G/(SU(2)K)$  corresponding to a highest root decomposition, is a symmetric space with a quaternionic Kähler structure, cf. [19]. Such manifolds are known as *Wolf spaces* and are listed in Besse [2, p. 409].

**Remark 2.9.** Via (2.3) the complexification of the tangent space of  $W(G)$  is isomorphic to  $S^1 \otimes V$ , so

$$(2.4) \quad \dim_{\mathbb{C}} V = \frac{1}{2} \dim_{\mathbb{R}} W(G).$$

**2.6. The Beauville Bundle.** A useful approach to the computation of the cohomogeneity of a nilpotent orbit  $\mathcal{O}$  is to consider the Beauville bundle  $\mathcal{N}(\mathcal{O})$  [1].  $\mathcal{N}(\mathcal{O})$  is a vector bundle that contains an open orbit which is identified with  $\mathcal{O}$ . Following Kobak & Swann [15] we shall outline a procedure for finding cohomogeneities.

Let  $\mathcal{O}$  be a nilpotent orbit in the semisimple Lie algebra  $\mathfrak{g}^{\mathbb{C}}$ . Let  $\sigma$  be the conjugation with respect to the compact real form  $\mathfrak{g}$ , and choose a standard triple  $\{H, X, Y = -\sigma X\}$  of a real  $\mathfrak{sl}(2, \mathbb{C})$  subalgebra with  $X \in \mathcal{O}$ . By the representation theory of  $\mathfrak{sl}(2, \mathbb{C})$  the eigenvalues of  $\text{ad}_H$  are integers. For each  $j \in \mathbb{N}$  we let  $\mathfrak{g}^{\mathbb{C}}(j)$  be the corresponding eigenspace of  $\text{ad}_H$  and set  $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}^{\mathbb{C}}(i)$  and  $\mathfrak{n} = \bigoplus_{i \geq 2} \mathfrak{g}^{\mathbb{C}}(i)$ . Then  $\mathfrak{p}$  is a parabolic subalgebra,  $\mathfrak{n}$  is a nilpotent subalgebra and  $X \in \mathfrak{n}$ . Let  $P$  be the corresponding parabolic subgroup  $P = \{g \in G^{\mathbb{C}} \mid \text{Ad}_g(\mathfrak{p}) = \mathfrak{p}\}$ . This is a closed complex Lie subgroup of  $G^{\mathbb{C}}$  with Lie algebra equal to  $\mathfrak{p}$ . In particular  $G^{\mathbb{C}}/P$  is a homogeneous space. Clearly  $\mathfrak{n}$  is an ideal of  $\mathfrak{p}$  and there is a natural action of  $P$  on  $G^{\mathbb{C}} \times \mathfrak{n}$ ,

$$p \cdot (g, N) \mapsto (gp^{-1}, \text{Ad}_p N), \quad p \in P, g \in G^{\mathbb{C}}, N \in \mathfrak{n}.$$

The *Beauville bundle*  $\mathcal{N}(\mathcal{O})$  is defined to be the vector bundle

$$\mathcal{N}(\mathcal{O}) = G^{\mathbb{C}} \times_P \mathfrak{n}$$

over  $G^{\mathbb{C}}/P$  with fibre  $\mathfrak{n}$ . Letting  $\pi$  denote the quotient map  $G^{\mathbb{C}} \times \mathfrak{n} \rightarrow \mathcal{N}(\mathcal{O})$  we have the following equality of stabilizers,  $G_{\pi(e, N)}^{\mathbb{C}} = P_N$ ,  $N \in \mathfrak{n}$ . If  $\tilde{\mathcal{O}}$  denotes the  $G^{\mathbb{C}}$ -orbit in  $\mathcal{N}(\mathcal{O})$  then

$$\mathcal{O} \cong G^{\mathbb{C}}/G_X^{\mathbb{C}} = G^{\mathbb{C}}/G_{\pi(e, X)}^{\mathbb{C}} \cong \tilde{\mathcal{O}}.$$

That is, we have a  $G^{\mathbb{C}}$ -equivariant isomorphism of the orbits  $\mathcal{O}$  and  $\tilde{\mathcal{O}}$ , defined by  $g \cdot X \mapsto g \cdot \pi(e, X)$ ,  $g \in G^{\mathbb{C}}$ .

Let  $\mathfrak{k} \subseteq \mathfrak{g}$  be the subalgebra satisfying  $\mathfrak{k}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}(0)$  and let  $K \subseteq G$  be the connected Lie subgroup with Lie algebra  $\mathfrak{k}$ . Now  $G^{\mathbb{C}}/P = G/K$  so  $\mathcal{N}(\mathcal{O}) = G \times_K \mathfrak{n}$ . By Theorem 2.1 the union of the principal orbits in the  $G$ -space  $\mathcal{N}(\mathcal{O})$  forms a dense set. Since  $\tilde{\mathcal{O}}$  is an open subset, a principal orbit of the  $G$ -space  $\tilde{\mathcal{O}}$  must be a principal orbit of the  $G$ -space  $\mathcal{N}(\mathcal{O})$ . Using the  $G$ -equivariant isomorphism we conclude

$$(2.5) \quad \text{cohom}_G \mathcal{O} = \text{cohom}_G \tilde{\mathcal{O}} = \text{cohom}_G \mathcal{N}(\mathcal{O}) = \text{cohom}_K \mathfrak{n}.$$

To find the cohomogeneity of  $\mathfrak{n}$  under the action of  $K$ , one merely has to use equation (2.1).

**Example 2.10.** We illustrate the *Beauville bundle method* by considering the nilpotent orbit  $\mathcal{O}$  in  $\mathfrak{sl}(n, \mathbb{C})$  with Jordan normal form  $(3, 1^{n-3})$ . For  $n \geq 4$  this is an orbit of height three. We take the



obvious choice of a standard triple (see Section 3.1) and obtain a  $K = U(1)_+ \times U(1)_- \times SU(n-2)$  representation

$$\mathfrak{n} = L_+^2 L_-^3 \oplus L_+ L_-^4 S^1 \mathbb{C}^{n-2} \oplus L_+ L_-^{-1} S^1 \mathbb{C}^{n-2}.$$

Fixing the first circle action, the problem is reduced to finding the cohomogeneity of the  $K_2 := U(1)SU(n-2)$  representation

$$\mathfrak{n}_2 := L^4 S^1 \mathbb{C}^{n-2} \oplus L^{-1} S^1 \mathbb{C}^{n-2},$$

Using the method described in Example 2.2 one finds that

$$\begin{aligned} \text{cohom}_{SU(n)} \mathcal{O} &= 1 + \text{cohom}_{K_2} \mathfrak{n}_2 = 3 + \text{cohom}_{SU(n-3)} S^1 \mathbb{C}^{n-3} \\ &= \begin{cases} 5 & n = 4 \\ 4 & n \geq 5 \end{cases}. \end{aligned}$$

For  $n = 3$ , the orbit (3) is the regular orbit of  $\mathfrak{sl}(3, \mathbb{C})$ . This is next-to-minimal and of cohomogeneity four (see [9]).

### 3. ORBITS OF HEIGHT THREE

By work of Dancer & Swann [9] all next-to-minimal orbits but one are of cohomogeneity two; the next-to-minimal orbit of  $\mathfrak{sl}(3, \mathbb{C})$  (the regular orbit) is of cohomogeneity four. In the view of Theorem 2.5, the nilpotent orbits of cohomogeneity three, if any, must therefore be of height three in the partial order. So indeed the orbits of Table 1 are our candidates in the search for orbits of cohomogeneity three (recall that  $A_3 = D_3$ ,  $B_2 = C_2$ ).

**3.1. Explicit Standard Triples.** The way of finding the cohomogeneity via the methods described in §2.6 is based upon the Jacobson-Morozov Theorem. Proposition 2.4 ensures the existence of a real  $\sigma$ -invariant  $\mathfrak{sl}(2, \mathbb{C})$  subalgebra associated to the nilpotent orbit  $\mathcal{O}$ . In this section we shall explicitly list the elements  $X$  of a standard triple  $\{\llbracket XY \rrbracket, X, Y\}$  with  $\sigma X = -Y$ , associated to nilpotent orbits of height three in classical Lie algebra.

Note that if  $\mathfrak{g}$  is an exceptional Lie algebra, the weighted Dynkin diagram provides the necessary information.

The complex simple Lie algebras of type  $B_{(n-1)/2}$  and  $D_{n/2}$  can be realized as the set  $\mathfrak{so}(n, \mathbb{C}) = \{Z \in \mathfrak{gl}(n, \mathbb{C}) | ZB + BZ^t = 0\}$  where  $B$  is a symmetric invertible matrix. The standard compact real form of  $\mathfrak{so}(n, \mathbb{C})$  is  $\sigma: Z \mapsto B\bar{Z}B = -\bar{Z}^t$ . The standard choice for  $B$  is the identity, but in this context we choose  $B$  differently. For orbits with Jordan normal form  $(2^6, 1^{n-12})$  we take

$$(3.1) \quad B = \begin{pmatrix} B_{12} & 0 \\ 0 & I_{n-12} \end{pmatrix}$$

Type	Height three orbit	Type	Height three orbit
$A_n$		$G_2$	$2\rightleftharpoons 0$
$n \geq 4$	$(3, 1^{n-2})$	$F_4$	$01\rightleftharpoons 00$
$n \geq 5$	$(2^3, 1^{n-5})$	$E_6$	$\overset{0}{00100}$
$B_n$		$E_7$	$\overset{0}{200000}, \overset{0}{000010}$
$n = 2$	$(5)$	$E_8$	$\overset{0}{0100000}$
$n \geq 3$	$(3, 2^2, 1^{2n-6})$		
$n \geq 6$	$(2^6, 1^{2n-11})$		
$C_n$			
$n \geq 3$	$(2^3, 1^{2n-12})$		
$D_n$			
$n = 3$	$(3^2)$		
$n \geq 4$	$(3, 2^2, 1^{2n-7})$		
$n \geq 6$	$(2^6, 1^{2n-12})$		

**TABLE 1.** Orbits of height three in simple Lie algebras.

where  $B_{12}$  is the  $(12 \times 12)$  matrix with 1's down the anti-diagonal and 0's elsewhere. In this picture we let

$$(3.2) \quad X = \text{diag}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, 0_{n-12}\right).$$

For the orbit in  $B_2$  with Jordan normal form (5),  $B$  is the  $(5 \times 5)$  matrix with 1's down the anti-diagonal, and our choice of  $X$  is

$$X = \begin{pmatrix} 0 & \sqrt{2}(1-i) & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{6}i & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6}i & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}(i-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the orbit with Jordan normal form  $(3^2)$  in  $D_3$ ,  $B$  is the  $(6 \times 6)$  matrix with 1's down the anti-diagonal, and

$$X = i\sqrt{2} \begin{pmatrix} \overset{0}{0} & \overset{1}{0} & \overset{0}{0} \\ \overset{0}{0} & \overset{0}{0} & \overset{1}{0} \\ \overset{0}{0} & \overset{0}{0} & \overset{0}{0} \\ & \overset{0}{0} & \overset{-1}{0} & \overset{0}{0} \\ & \overset{0}{0} & \overset{0}{0} & \overset{-1}{0} \\ & \overset{0}{0} & \overset{0}{0} & \overset{0}{0} \end{pmatrix}.$$

When considering the orbit in  $B_3$  characterized by the Jordan type  $(3, 2^2)$  we choose

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_3 \\ 0 & I_3 & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & 0 & v^t \\ -v & 0 & A \\ 0 & 0 & 0 \end{pmatrix},$$

where  $v = (\sqrt{2}, 0, 0)$  and  $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ . By adding trivial blocks, as in (3.1) and (3.2), this is easily generalized to the orbits  $(3, 2^2, 1^{n-7})$  in  $B_{(n-1)/2}, D_{n/2}$ .

For type  $A_n = \mathfrak{sl}(n+1, \mathbb{C})$  there are two orbits to consider, characterized by the partitions  $(2^3, 1^{n-5})$  and  $(3, 1^{n-2})$ . Suitable choices of  $X$  are evident,

$$X = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix},$$

respectively. These have the desired properties with respect to the standard compact real form  $\sigma: Z \mapsto -\overline{Z}^t$ .

For type  $C_n = \mathfrak{sp}(n, \mathbb{C})$  there is one orbit to consider. We take

$$X = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ & & & 0 \end{pmatrix}.$$

**3.2. The Cohomogeneities.** With the choices of the previous section we are able to find the centralizer  $\mathfrak{t}$  and representation  $\mathfrak{n}$  of §2.6 in each case. This leads to Tables 2 & 3 which exhaust the complete list of adjoint nilpotent orbits of cohomogeneity three in simple Lie algebras.

**3.3. Examples of Standard Forms.** With two exceptions, a nilpotent orbit of cohomogeneity three lies in a classical Lie algebra. It is either described by the partition  $(2^3, 1^k)$  when the Lie algebra is of type  $A$  or  $C$ , or by  $(2^6, 1^k)$  if the Lie algebra has type  $B$  or  $D$ . For these orbits, as the following examples show, it is fairly simple to ‘diagonalize’ elements of  $\mathfrak{n}$  via the action of  $K$ . This in fact only involves the spectral theorem applied to Hermitian matrices.

**Example 3.1.** Consider the following action of the unitary group

$$\begin{aligned} U(n) \times \text{Sym}(n, \mathbb{C}) &\rightarrow \text{Sym}(n, \mathbb{C}), \\ (g, Z) &\mapsto g \cdot Z := gZg^t. \end{aligned}$$

Fix an element  $Z$  in  $\text{Sym}(n, \mathbb{C})$ . Then the Hermitian matrix  $Z\overline{Z}$  has real non-negative eigenvalues  $\mu_1 \geq \dots \geq \mu_n \geq 0$  and is diagonalized by some  $g \in U(n)$  via conjugation,

$$g(Z\overline{Z})\overline{g}^t = \text{diag}(\mu_1, \dots, \mu_n) = D.$$

Type\Orbit	$\mathfrak{k}\backslash\mathfrak{n}$	$\text{cohom}_G \mathcal{O}$
$A_5$ ( $2^3$ )	$\mathbb{R} + \mathfrak{su}(3)_a + \mathfrak{su}(3)_b$ $L \text{ End}(\mathbb{C}^3)$	3
$A_n$ ( $n>5$ ) ( $2^3, 1^{n-5}$ )	$\mathbb{R}_+ + \mathfrak{su}(3)_a + \mathfrak{su}(3)_b + \mathbb{R}_- + \mathfrak{su}(n-5)$ $L_+ \text{ End}(\mathbb{C}^3)$	3
$A_n$ ( $n>3$ ) ( $3, 1^{n-2}$ )	$\mathbb{R}_+ + \mathbb{R}_- + \mathfrak{su}(n-1)$ $L_+^2 L_-^3 + L_+(L_-^4 + L_-^{-1}) S^1 \mathbb{C}^{n-1}$	4
$B_2$ (5)	$\mathbb{R}_+ + \mathbb{R}_-$ $(L_+ + L_+^{-1}) L_- + L_+ S^1 \mathbb{C}^2$	6
$B_3$ ( $3, 2^2$ )	$\mathbb{R}_+ + \mathbb{R}_- + \mathfrak{su}(2)$ $L_+ + L_-^2 + L_+ L_- S^1 \mathbb{C}^2$	3
$B_n$ ( $n>3$ ) ( $3, 2^2, 1^{2n-6}$ )	$\mathbb{R}_+ + \mathbb{R}_- + \mathfrak{su}(2) + \mathfrak{so}(n)$ $L_-^2 + L_+(L_- S^1 \mathbb{C}^2 + S^1 \mathbb{R}^n)$	4
$B_n$ ( $n>5$ ) ( $2^6, 1^{2n-11}$ )	$\mathbb{R} + \mathfrak{su}(6) + \mathfrak{so}(2n-11)$ $L \Lambda^2 \mathbb{C}^6$	3
$C_3$ ( $2^3$ )	$\mathbb{R} + \mathfrak{su}(3)$ $L S^2 \mathbb{C}^3$	3
$C_n$ ( $n>3$ ) ( $2^3, 1^{2n-6}$ )	$\mathbb{R} + \mathfrak{su}(3) + \mathfrak{sp}(n-3)$ $L S^2 \mathbb{C}^3$	3
$D_3$ ( $3^2$ )	$\mathbb{R}_+ + \mathbb{R}_- + \mathfrak{su}(2)$ $L_-^8 + (L_+ + L_+^{-1}) L_- S^1 \mathbb{C}^2$	5
$D_n$ ( $n>3$ ) ( $3, 2^2, 1^{2n-7}$ )	$\mathbb{R}_+ + \mathbb{R}_- + \mathfrak{su}(2) + \mathfrak{so}(n)$ $L_-^2 + L_+(L_- S^1 \mathbb{C}^2 + S^1 \mathbb{R}^n)$	4
$D_n$ ( $n>5$ ) ( $2^6, 1^{2n-12}$ )	$\mathbb{R} + \mathfrak{su}(6) + \mathfrak{so}(2n-12)$ $L \Lambda^2 \mathbb{C}^6$	3

**TABLE 2.** Beauville-data of orbits of height three in classical Lie algebras.  $\mathfrak{n}$  is decomposed as a  $K$ -module,  $\text{End}(\mathbb{C}^3)$  is an  $SU(3)_a SU(3)_b$ -module under the action  $(A, B)X \mapsto AXB^{-1}$ .

Define the symmetric matrix  $X = (x_{ij})$  by  $X = g \cdot Z$  and observe that  $X\bar{X} = D$ . Thus,

$$\sum_{k=1}^n |x_{ik}|^2 = \mu_i \quad \text{and} \quad \sum_{k=1}^n x_{ik} \bar{x}_{kj} = 0 \quad \text{for } i \neq j.$$

Type\Orbit	$\mathfrak{k}\backslash\mathfrak{n}$	$\mathrm{cohom}_G \mathcal{O}$
$G_2$	$\mathbb{R} + \mathfrak{su}(2)$	6
$2 \Rightarrow 0$	$L + LS^3\mathbb{C}^2$	
$F_4$	$\mathbb{R} + \mathfrak{su}(2) + \mathfrak{su}(3)$	4
$01 \Rightarrow 00$	$L^3 S^1\mathbb{C}^2 + L^2 S^2\mathbb{C}^3$	
$E_6$	$\mathbb{R} + \mathfrak{su}(2) + \mathfrak{su}(3)_a + \mathfrak{su}(3)_b$	4
$00100$	$L^3 S^1\mathbb{C}^2 + L^2 \mathrm{End}(\mathbb{C}^3)$	
$E_7$	$\mathbb{R} + \mathfrak{e}_6$	3
$200000$	$LV_{27}$	
$E_7$	$\mathbb{R} + \mathfrak{su}(2) + \mathfrak{su}(6)$	4
$000010$	$L^3 S^1\mathbb{C}^2 + L^2 \Lambda^2\mathbb{C}^6$	
$E_8$	$\mathbb{R} + \mathfrak{su}(2) + \mathfrak{e}_6$	4
$0100000$	$L^2 S^1\mathbb{C}^2 + L^3 V_{27}$	

**TABLE 3.** Beauville-data of orbits of height three in exceptional Lie algebras.  $V_{27}$  is a 27-dimensional irreducible  $E_6$ -module.

Making use of the symmetry  $x_{ij} = x_{ji}$  it follows that

(3.3)

$$\begin{aligned}
x_{pq}(\mu_p - \mu_q) &= \sum_{j \neq q} x_{qp} |x_{pj}|^2 - \sum_{j \neq p} x_{pq} |x_{qj}|^2 + (x_{qp} |x_{pq}|^2 - x_{pq} |x_{qp}|^2) \\
&= \sum_{j \neq q} \left( \sum_k x_{qk} \bar{x}_{kj} \right) x_{pj} - \sum_{j \neq p} \left( \sum_k x_{pk} \bar{x}_{kj} \right) x_{qj} \\
&= 0.
\end{aligned}$$

If the eigenvalues are all different,  $\mu_1 > \dots > \mu_n$ , this implies that  $X$  is diagonal. The general case follows immediately: let  $M$  be the dense subset of  $\mathrm{Sym}(n, \mathbb{C})$  whose elements have different eigenvalues and find a sequence  $(X_i) \subset M$  such that  $X_i \rightarrow X$ . Then there exists elements  $g_i \in U(n)$  such that  $g_i (X_i \bar{X}_i) \bar{g}_i^t$  is diagonal. By compactness we may assume that  $g_i \rightarrow g_0 \in U(n)$ . Then  $g_0 (X \bar{X}) \bar{g}_0^t$  is diagonal with non-equal eigenvalues and (3.3) implies that  $g_0 \cdot X = (g_0 g) \cdot Z$  is diagonal.

**Example 3.2.** In this example the symmetric matrices are replaced by skew-symmetric matrices,

$$\begin{aligned}
U(n) \times \Lambda^2 \mathbb{C}^n &\rightarrow \Lambda^2 \mathbb{C}^n, \\
(g, Z) &\mapsto g \cdot Z := gZg^t.
\end{aligned}$$

Fix  $Z \in \mathfrak{so}(n, \mathbb{C}) = \Lambda^2 \mathbb{C}^n = \{Z \in \mathfrak{gl}(n, \mathbb{C}) | Z + Z^t = 0\}$  with  $n$  even (the case for  $n$  odd is similar). The eigenvalues of the Hermitian matrix  $Z\bar{Z}$  are real and non-positive and there exists a  $g \in U(n)$  with

$$g(Z\bar{Z})\bar{g}^t = \text{diag}(\tilde{\mu}_1, \dots, \tilde{\mu}_n) = D \quad \text{where} \quad 0 \geq \tilde{\mu}_1 \geq \dots \geq \tilde{\mu}_n.$$

Put  $X = g \cdot Z$  and assume  $X$  to have  $n$  different eigenvalues. Then  $X$  is diagonalizable and  $\mathbb{C}^n$  is the direct sum of eigenspaces for both of the commuting matrices  $X$  and  $\bar{X}$ . As  $X$  is skew-symmetric it follows that  $\bar{X}v = -\bar{\lambda}v$  whenever  $v$  is a  $\lambda$ -eigenvector of  $X$ . Thus  $v$  and  $\bar{v}$  are eigenvectors of  $X\bar{X}$  corresponding to the same eigenvalue  $-|\lambda|^2$ . Consequently,

$$(3.4) \quad X\bar{X} = \text{diag}(\mu_1, \mu_1, \mu_2, \mu_2, \dots, \mu_{n/2}, \mu_{n/2}), \quad \mu_1 > \dots > \mu_{n/2}.$$

By a computation similar to (3.3) one gets that

$$x_{ij}(\mu_i - \mu_j) = 0,$$

so  $X = g_0 \cdot Z$  has the block-diagonal form

$$(3.5) \quad \text{diag}\left(\begin{pmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & x_{n-1,n} \\ -x_{n-1,n} & 0 \end{pmatrix}\right).$$

Notice that the collection  $M$  of elements in  $\mathfrak{so}(n, \mathbb{C})$  with  $n$  different eigenvalues forms a dense subset of  $\mathfrak{so}(n, \mathbb{C})$ . Thus, if  $X$  does not have  $n$  different eigenvalues we may find a sequence  $X_i \in M$  such that  $X_i \rightarrow X$ . The previous argument implies that there exist  $h_i \in U(n)$  such that  $h_i(X_i\bar{X}_i)\bar{h}_i^t$  has the form in (3.4) and  $h_i \cdot X$  is given by (3.5) for all  $i$ . Since  $U(n)$  is compact we may demand  $h_i \rightarrow h_0 \in U(n)$ . Now  $(h_0g) \cdot Z = h_0 \cdot X$  is in the form given by (3.5).

**Example 3.3.** Let  $A \in M(n, \mathbb{C})$ . The self-adjoint matrix  $\bar{A}^t A$  is diagonalized by some  $q \in SU(n)$ ,  $\bar{q}^t(\bar{A}^t A)q = \text{diag}(\lambda_1, \dots, \lambda_n) := D$  and  $\lambda_i \geq 0$ . The  $i$ 'th column  $q_i$  of  $q$  is an eigenvector of  $A$  with eigenvalue  $\lambda_i$ . If we put  $u_i = \lambda_i^{-1/2} A q_i$  then  $u := (u_1 \dots u_n) \in U(n)$  and  $Aq = u\sqrt{D}$ . Thus any  $n \times n$  matrix can be diagonalized by the action

$$(U(n) \times SU(n)) \times M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C}), \\ ((u, q), A) \mapsto u A \bar{q}^t.$$

Notice that this is actually the singular value decomposition.

#### 4. POTENTIALS FOR HYPERKÄHLER METRICS IN COHOMOGENEITY THREE

In this chapter  $\mathfrak{g}^c$  is simple and  $\mathcal{O}$  is a nilpotent orbit of cohomogeneity three under the action of the compact group  $G$ . Our aim is to find  $G$ -invariant hyperKähler metrics with Kähler potential on the

complex symplectic manifold  $(\mathcal{O}, I, \Sigma)$  where  $I$  is the natural complex structure and  $\Sigma$  is the Kirillov-Kostant-Souriau form.

Recall that a *hyperKähler potential*  $\rho: \mathcal{O} \rightarrow \mathbb{R}$  is a function that is a simultaneous Kähler potential for each of the complex structures, i.e.  $\omega_I = -i\partial_I\bar{\partial}_I\rho$  etc. For our purposes it is convenient to note that  $i\partial_I\bar{\partial}_I\rho = \frac{1}{2}id(d - iId)\rho = \frac{1}{2}dId\rho$ . In general, the existence of a global Kähler potential is a very delicate matter. Indeed a compact manifold admits no global Kähler potential. However, a local potential always exists (see eg. [10]). There is no direct analogue of this for hyperKähler potentials. Generally, even the existence of a local hyperKähler potential may not be possible, see Swann [19].

In this section we use the following abbreviation for the conjugation corresponding to the compact real form

$$Z' = \sigma Z, \quad Z \in \mathfrak{g}^{\mathbb{C}}.$$

**4.1. Kähler Potentials Depending on Three Invariants.** Define three functions on the nilpotent orbit  $\mathcal{O}$

$$\begin{aligned} \eta_1(X) &= \langle X, X' \rangle, \quad \eta_2(X) = -\langle [XX'], [XX'] \rangle \quad \text{and} \\ \eta_3(X) &= -\langle [XX'X'], [X'XX'] \rangle, \quad X \in \mathcal{O}, \end{aligned}$$

Observe that  $\eta_2(X) = \eta_1([XX'])$  and  $\eta_3(X) = \eta_1([XX'X'])$ , thus  $\eta_1, \eta_2, \eta_3 : \mathcal{O} \rightarrow \mathbb{R}$  are positive  $G$ -invariant functions. Assume that  $\rho : \mathcal{O} \rightarrow \mathbb{R}$  is a  $G$ -invariant Kähler potential for  $(\mathcal{O}, I)$ . The fact that the group  $G$  acts with cohomogeneity three allows us to assume  $\rho$  to be a function of our three invariant functions:  $\rho = \rho(\eta_1, \eta_2, \eta_3)$ . The Kähler form is the second derivative of  $\rho$ , so by differentiation one may find an explicit expression of  $\omega_I$ . To ease the notation let  $\mathcal{Z} : \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$  be the map

$$\mathcal{Z}X = [XX'X'X'X] + [X'X'XXX'] + 2[X'XX'XX'].$$

**Lemma 4.1.** *The Kähler form  $\omega_I$  defined by  $\rho$  is*

$$\begin{aligned}
(4.1) \quad & \omega_I(\xi_A, \xi_B)_X = 2\rho_1 \operatorname{Im} \langle \xi_A, \xi_B' \rangle \\
& - 4\rho_2 \operatorname{Im} \langle \xi_A, 2\llbracket X'X\xi_B' \rrbracket - \llbracket XX'\xi_B' \rrbracket \rangle \\
& - 2\rho_3 \operatorname{Im} \{ -2 \langle \xi_A, \llbracket XXX'X'\xi_B' \rrbracket \rangle + 3 \langle \xi_A, \llbracket XX'XX'\xi_B' \rrbracket \rangle \\
& \quad + 3 \langle \xi_A, \llbracket XX'X'X\xi_B' \rrbracket \rangle + 3 \langle \xi_A, \llbracket X'XXX'\xi_B' \rrbracket \rangle \\
& \quad - 12 \langle \xi_A, \llbracket X'XX'X\xi_B' \rrbracket \rangle + 3 \langle \xi_A, \llbracket X'X'XX\xi_B' \rrbracket \rangle \} \\
& + 2\rho_{11} \operatorname{Im} (\langle \xi_A, X' \rangle \langle \xi_B', X \rangle) \\
& - 4\rho_{12} \operatorname{Im} (\langle \xi_A, X' \rangle \langle \xi_B', \llbracket XX'X \rrbracket \rangle + \langle \xi_A, \llbracket X'XX' \rrbracket \rangle \langle \xi_B', X \rangle) \\
& + 2\rho_{13} \operatorname{Im} (\langle \xi_A, X' \rangle \langle \xi_B', \mathcal{Z}X' \rangle + \langle \xi_A, \mathcal{Z}X \rangle \langle \xi_B', X \rangle) \\
& + 8\rho_{22} \operatorname{Im} (\langle \xi_A, \llbracket X'XX' \rrbracket \rangle \langle \xi_B', \llbracket XX'X \rrbracket \rangle) \\
& - 4\rho_{23} \operatorname{Im} (\langle \xi_A, \llbracket X'XX' \rrbracket \rangle \langle \xi_B', \mathcal{Z}X' \rangle \\
& \quad + \langle \xi_A, \mathcal{Z}X \rangle \langle \xi_B', \llbracket XX'X \rrbracket \rangle) \\
& + 2\rho_{33} \operatorname{Im} (\langle \xi_A, \mathcal{Z}X \rangle \langle \xi_B', \mathcal{Z}X' \rangle),
\end{aligned}$$

for any  $\xi_A, \xi_B \in \Gamma(T\mathcal{O})$  and  $X \in \mathcal{O}$ .

*Proof.* We start by expressing  $\omega_I$  in terms of the derivatives of the invariant functions  $\eta_j$ ,  $j = 1, 2, 3$ ,

$$\begin{aligned}
(4.2) \quad & -2\omega_I = dId\rho \\
& = \rho_1 dId\eta_1 + \rho_2 dId\eta_2 + \rho_3 dId\eta_3 \\
& \quad + \rho_{11} d\eta_1 \wedge Id\eta_1 + \rho_{22} d\eta_2 \wedge Id\eta_2 + \rho_{33} d\eta_3 \wedge Id\eta_3 \\
& \quad + \rho_{12} (d\eta_1 \wedge Id\eta_2 + d\eta_2 \wedge Id\eta_1) \\
& \quad + \rho_{13} (d\eta_1 \wedge Id\eta_3 + d\eta_3 \wedge Id\eta_1) \\
& \quad + \rho_{23} (d\eta_2 \wedge Id\eta_3 + d\eta_3 \wedge Id\eta_2).
\end{aligned}$$

The next step is to determine all differentials  $d\eta_j$  and  $dId\eta_j$ ,  $j = 1, 2, 3$ . However, only calculations involving  $\eta_3$  will be carried out in detail, the other cases being similar if not simpler. Let us write  $\eta_3 = \eta_1 \circ \phi$  where  $\phi$  is the endomorphism of  $\mathfrak{g}^c$  defined by  $\phi(Z) = \llbracket ZZZ' \rrbracket$ ,  $Z \in \mathfrak{g}^c$ . Differentiating  $\phi$  and  $\eta_1$  is easy,

$$\begin{aligned}
(d\eta_1)_{\phi(X)}(Z) &= \langle Z, \llbracket X'X'X \rrbracket \rangle + \langle \llbracket XXX' \rrbracket, Z' \rangle, \quad Z \in \mathfrak{g} \\
d\phi(\xi_A)_X &= \llbracket \xi_A XX' \rrbracket + \llbracket X\xi_A X' \rrbracket + \llbracket XX\xi_A' \rrbracket.
\end{aligned}$$



A repeated use of the Jacobi identity and the ad-invariance of the Killing form give us the derivative of  $\eta_3$

$$\begin{aligned} d\eta_3(\xi_A)_X &= \langle \xi_A, [[X X'] X' X' X] - [X' X X' X' X] + [X' X' X X X'] \rangle \\ &\quad + \langle \xi_A', [X X X' X' X] + [[X' X] X X X'] - [X X' X X X'] \rangle \\ &= 2 \operatorname{Re} \sum_{j=1}^3 \langle \xi_A, \mathcal{Z}_j X \rangle, \end{aligned}$$

Here we have defined  $\mathcal{Z}_1 X = [X X' X' X' X]$ ,  $\mathcal{Z}_2 X = [X' X' X X X']$  and  $\mathcal{Z}_3 X = 2 [X' X X' X X']$ . Therefore  $Id\eta_3(\xi_A)_X = 2 \operatorname{Im} \sum_j \langle \xi_A, \mathcal{Z}_j X \rangle$ , of which we are now able to find the derivative. However, this is somewhat more demanding. First observe that

$$\begin{aligned} (4.3) \quad dId\eta_3(\xi_A, \xi_B)_X &= \xi_A|_X(Y \mapsto Id\eta_3((\xi_B)_Y)) - \xi_B|_X(Y \mapsto Id\eta_3((\xi_A)_Y)) - Id\eta_3([ \xi_A \xi_B ])_X \\ &= 2 \operatorname{Im} \sum_{j=1}^3 \{ \xi_A|_X(Y \mapsto \langle [BY], \mathcal{Z}_j Y \rangle) - \xi_B|_X(Y \mapsto \langle [AY], \mathcal{Z}_j Y \rangle) \\ &\quad - \langle [XAB], \mathcal{Z}_j X \rangle \}. \end{aligned}$$

Letting  $\chi_j$  denote the summands in (4.3) involving  $\mathcal{Z}_j$  one finds that

$$\begin{aligned} \chi_1 &= 2 \operatorname{Im} \langle \xi_A, -[\xi_B X' X' X' X] + [[\xi_B' X'] X X X'] - [[X \xi_B' X'] X X'] \\ &\quad + [X' X X X' \xi_A'] + [X' X' X' X \xi_B] - [\xi_B X' X' X' X] - [X \xi_B' X' X' X] \\ &\quad - [X X' \xi_B' X' X] + [X X' X' X \xi_B'] - [X X' X' X' \xi_B] \rangle. \end{aligned}$$

For reasons we shall explain later, it is desirable to place the  $\xi_B$ 's in the above formula to the far right. Although the calculations are lengthy when using the Jacobi identity, it is a straight forward exercise. One finds that

$$\begin{aligned} \chi_1 &= 2 \operatorname{Im} \langle \xi_A, 2 [X X X' X' \xi_B'] - 6 [X X' X X' \xi_B'] + 3 [X X' X' X \xi_B'] \\ &\quad - 3 [X X' X' X' \xi_B] + 3 [X' X X X' \xi_B'] + 6 [X' X X' X' \xi_B] \\ &\quad - 6 [X' X' X X' \xi_B] + 3 [X' X' X' X \xi_B] \rangle. \end{aligned}$$

Similarly, the expressions for the remaining two summands are

$$\begin{aligned} \chi_2 &= 2 \operatorname{Im} \langle \xi_A, -2 [X X X' X' \xi_B'] + 4 [X X' X X' \xi_B'] - 8 [X' X X' X \xi_B'] \\ &\quad + [X X' X' X' \xi_B] + [X' X X X' \xi_B'] + [X X' X' X \xi_B'] - [X' X' X' X \xi_B] \\ &\quad + 2 [X' X' X X \xi_B'] + 2 [X' X' X X' \xi_B] - 2 [X' X X' X' \xi_B] \rangle \end{aligned}$$

and

$$\begin{aligned} \chi_3 = & -4 \operatorname{Im} \langle \xi_A, 2 \llbracket XXX'X'\xi_B' \rrbracket - 4 \llbracket XX'XX'\xi_B' \rrbracket + 8 \llbracket X'XX'X\xi_B' \rrbracket \\ & - \llbracket XX'X'X'\xi_B \rrbracket - \llbracket X'XXX'\xi_B' \rrbracket - \llbracket XX'X'X\xi_B' \rrbracket - \llbracket X'X'X'X\xi_B \rrbracket \\ & - 2 \llbracket X'X'XX\xi_B' \rrbracket - 2 \llbracket X'X'XX'\xi_B \rrbracket + 2 \llbracket X'XX'X'\xi_B \rrbracket \rangle. \end{aligned}$$

We are now in the position to write down the second derivative of  $\eta_3$  in a somewhat shortened form,

$$\begin{aligned} dId\eta_3(\xi_A, \xi_B)_X = & 12 \operatorname{Im} \langle \xi_A, -\frac{2}{3} \llbracket XXX'X'\xi_B' \rrbracket - 4 \llbracket X'XX'X\xi_B' \rrbracket \\ & + \llbracket XX'X'X\xi_B' \rrbracket + \llbracket X'XXX'\xi_B' \rrbracket + \llbracket XX'XX'\xi_B' \rrbracket + \llbracket X'X'XX\xi_B' \rrbracket \rangle. \end{aligned}$$

As already noted, the determination of  $d\eta_j$ ,  $dId\eta_j$ ,  $j = 1, 2$  represents a similar exercise, moreover it is already present in [15]

$$\begin{aligned} d\eta_1(\xi_A)_X = & 2 \operatorname{Re} \langle \xi_A, X' \rangle, \quad dId\eta_1(\xi_A, \xi_B)_X = -4 \operatorname{Im} \langle \xi_A, \xi_B' \rangle, \\ d\eta_2(\xi_A)_X = & -4 \operatorname{Re} \langle \xi_A, \llbracket X'XX' \rrbracket \rangle \quad \text{and} \\ dId\eta_2(\xi_A, \xi_B)_X = & 8 \operatorname{Im} \langle \xi_A, 2 \llbracket X'X\xi_B' \rrbracket - \llbracket XX'\xi_B' \rrbracket \rangle. \end{aligned}$$

To finish, note that with  $\mathcal{Z}X = \sum_{j=1}^3 \mathcal{Z}_j X$  we have

$$\begin{aligned} (d\eta_1 \wedge Id\eta_3 + d\eta_3 \wedge Id\eta_1)(\xi_A, \xi_B)_X = & 4 (\operatorname{Re} \langle \xi_A, X' \rangle \operatorname{Im} \langle \xi_B, \mathcal{Z}X \rangle - \operatorname{Re} \langle \xi_B, X' \rangle \operatorname{Im} \langle \xi_A, \mathcal{Z}X \rangle \\ & + \operatorname{Re} \langle \xi_A, \mathcal{Z}X \rangle \operatorname{Im} \langle \xi_B, X' \rangle - \operatorname{Re} \langle \xi_B, \mathcal{Z}X \rangle \operatorname{Im} \langle \xi_A, X' \rangle) \\ = & -4 \operatorname{Im} (\langle \xi_A, X' \rangle \langle \xi_B', \mathcal{Z}X \rangle + \langle \xi_A, \mathcal{Z}X \rangle \langle \xi_B', X' \rangle), \end{aligned}$$

with similar expressions for the other summands of (4.2). Collecting the relevant formulæ completes the proof.  $\square$

Supposing  $g(\cdot, \cdot) = \omega_I(I\cdot, \cdot)$  to be non-degenerate we may define an endomorphism  $J$  of the tangent bundle  $T\mathcal{O}$  by

$$(4.4) \quad g(\xi_A, \xi_B) = \operatorname{Re} \Sigma(J\xi_A, \xi_B)$$

**Lemma 4.2.** *The endomorphism  $J$  is given by*

$$\begin{aligned}
(J\xi_A)_X &= -2\rho_1 \llbracket X\xi_A' \rrbracket \\
&+ 4\rho_2 (2 \llbracket XX'X\xi_A' \rrbracket - \llbracket XXX'\xi_A' \rrbracket) \\
&+ 2\rho_3 ( - 2 \llbracket XXXX'X'\xi_A' \rrbracket + 3 \llbracket XXX'XX'\xi_A' \rrbracket \\
&\quad + 3 \llbracket XXX'X'X\xi_A' \rrbracket + 3 \llbracket XX'X'XX\xi_A' \rrbracket \\
&\quad - 12 \llbracket XX'XX'X\xi_A' \rrbracket + 3 \llbracket XX'XXX'\xi_A' \rrbracket ) \\
(4.5) \quad &- 2\rho_{11} \langle \xi_A', X \rangle \llbracket XX' \rrbracket \\
&+ 4\rho_{12} ( \langle \xi_A', X \rangle \llbracket XX'XX' \rrbracket + \langle \xi_A', \llbracket XX'X \rrbracket \rangle \llbracket XX' \rrbracket ) \\
&- 2\rho_{13} ( \langle \xi_A', X \rangle \llbracket X(\mathcal{Z}X) \rrbracket + \langle \xi_A', \mathcal{Z}X' \rangle \llbracket XX' \rrbracket ) \\
&- 8\rho_{22} \langle \xi_A', \llbracket XX'X \rrbracket \rangle \llbracket XX'XX' \rrbracket \\
&+ 4\rho_{23} ( \langle \xi_A', \llbracket XX'X \rrbracket \rangle \llbracket X(\mathcal{Z}X) \rrbracket + \langle \xi_A', \mathcal{Z}X' \rangle \llbracket XX'XX' \rrbracket ) \\
&- 2\rho_{33} \langle \xi_A', \mathcal{Z}X' \rangle \llbracket X(\mathcal{Z}X) \rrbracket,
\end{aligned}$$

for any  $\xi_A, \xi_B \in \Gamma(T\mathcal{O})$  and  $X \in \mathcal{O}$ .

*Proof.*  $J$  is determined by the relation

$$- \operatorname{Re} \langle J_X \xi_A, B \rangle = \omega_I(I\xi_A, \xi_B)_X.$$

and the non-degeneracy of the form  $\langle \cdot, \cdot \rangle$ . As  $I\xi_A = i\xi_A$ , the right-hand side above equals the right-hand side of equation (4.1) with the slight modification of replacing ‘Im’ with ‘Re’ throughout. The result can now be read off equation (4.1).  $\square$

*Remark 4.3.* Notice that  $J\xi_A$  remains in the subalgebra generated by  $\{X, X', A, A'\}$ .

**4.2. Generic Cohomogeneity-Three Orbits.** For any  $X \in \mathcal{O}$ , assume the Lie span of  $\{X, X'\}$  to be embedded into a  $\sigma$ -invariant subalgebra isomorphic to three copies of  $\mathfrak{sl}(2, \mathbb{C})$ ; we shall denote such an algebra by  $\mathfrak{sl}(2, \mathbb{C})^3$ .

*Remark 4.4.* The majority of cohomogeneity-three orbits satisfy the above assumption, and as such will be referred to as *generic cohomogeneity three orbits*. In fact it shall become clear that  $\mathfrak{so}(7, \mathbb{C})$  possesses the only *special cohomogeneity three orbit* (see § 4.2.4). The orbits of cohomogeneity three are listed in Table 4.

The  $\sigma$ -invariance of  $\mathfrak{sl}(2, \mathbb{C})^3$  tell us that the restriction  $\tilde{\sigma} := \sigma|_{\mathfrak{sl}(2, \mathbb{C})^3}$  is a compact real form of  $\mathfrak{sl}(2, \mathbb{C})^3$ , so the +1 eigenspace of  $\tilde{\sigma}$  is isomorphic to three copies of  $\mathfrak{su}(2)$ ,

$$(\mathfrak{sl}(2, \mathbb{C})^3)^{\tilde{\sigma}} = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_0 \oplus \mathfrak{su}(2)_-.$$

Generic		Special	
Type	Orbit	Type	Orbit
$A_n$	$(2^3, 1^{n-5})$	$B_3$	$(3, 2^2)$
$B_{(n-1)/2}, D_{n/2}$	$(2^6, 1^{n-12})$		
$C_n$	$(2^3, 1^{2n-6})$		
$E_7$	$\begin{smallmatrix} 0 \\ 200000 \end{smallmatrix}$		

**TABLE 4.** Orbits of cohomogeneity three in simple Lie algebras

Let  $\mathfrak{sl}(2, \mathbb{C})_\delta$  denote the complexification of  $\mathfrak{su}(2)_\delta$ , where  $\delta$  refers to any of the three subscripts  $\{+, 0, -\}$ . Thus

$$X \in \mathfrak{sl}(2, \mathbb{C})^3 = \mathfrak{sl}(2, \mathbb{C})_+ \oplus \mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_- \subset \mathfrak{g}^{\mathbb{C}}.$$

We write  $X = X_+ + X_0 + X_-$ ,  $X_\delta \in \mathfrak{sl}(2, \mathbb{C})_\delta$  and assume  $X$  to be generic (i.e. each  $X_\delta$  is nonzero). Notice that  $\mathcal{O} \cap \mathfrak{sl}(2, \mathbb{C})_\delta$  is the minimal orbit of  $\mathfrak{sl}(2, \mathbb{C})_\delta$  and is of cohomogeneity one under the action of  $SU(2)_\delta$ . It follows from Remark 2.3 that

$$(4.6) \quad Z_{abc} = aX_+ + bX_0 + cX_- \in \mathcal{O}$$

for all  $a, b, c \in \mathbb{R}_+$ . The element  $Z = aX_+ + bX_0$  is the limit of (4.6) as  $c$  approaches zero so it lies in the closure of  $\mathcal{O}$ . Considered as endomorphisms of  $\mathfrak{sl}(2, \mathbb{C})^3$  the rank of  $\text{ad}_Z$  is obviously strictly less than the rank of  $\text{ad}_{Z_{abc}}$ , since the three subalgebras commute. On the other hand, the orthogonal complement of  $\mathfrak{sl}(2, \mathbb{C})^3$  is invariant under the adjoint action of  $\mathfrak{sl}(2, \mathbb{C})^3$ . Since  $Z_{abc}$  is an element of  $\mathcal{O}$  the rank of  $\text{ad}_{Z_{abc}}$  equals the constant rank of  $\text{ad}_X$  on  $\mathfrak{sl}(2, \mathbb{C})^3$  and its orthogonal complement for all  $a, b, c \in \mathbb{R}_+$ . By continuity the rank of  $\text{ad}_Z$  cannot exceed the rank of  $\text{ad}_{Z_{abc}}$  on the orthogonal complement. Taken together,  $Z$  cannot be an element of  $\mathcal{O}$ , and via (2.2) therefore lies in an orbit of height 2 or 1. Repeating this argument we find that  $X_\delta$  lies in the minimal orbit of  $\mathfrak{g}^{\mathbb{C}}$  and is conjugate to a highest root vector.

Expressing everything in terms of  $G$ -invariant objects, we may restrict our attention to points of the form

$$(4.7) \quad X = X_+ + X_0 + X_- = sE_+ + rE_0 + tE_-,$$

where  $s, r, t > 0$  and each  $E_\delta$  is a highest root vector in  $\mathfrak{g}^{\mathbb{C}}$ . Choose  $F_\delta = -E'_\delta$ ,  $H_\delta = \llbracket E_\delta F_\delta \rrbracket$ , so that  $\{H_\delta, E_\delta, F_\delta\}$  is a standard basis of  $\mathfrak{sl}(2, \mathbb{C})_\delta$ . Let  $\langle \cdot, \cdot \rangle_\delta$  denote the negative of the Killing form of  $\mathfrak{sl}(2, \mathbb{C})_\delta$ . By Schur's Lemma we have  $\langle \cdot, \cdot \rangle|_{\mathfrak{sl}(2, \mathbb{C})_\delta} = k_\delta^2 \langle \cdot, \cdot \rangle_\delta$ , where the constant  $k_\delta^2$  is

strictly positive because  $\mathfrak{su}(2)_\delta \subset \mathfrak{g}$ . The algebras  $\mathfrak{sl}(2, \mathbb{C})_\delta$  correspond each to a highest root in the same semisimple Lie algebra, so they are conjugate by the Weyl group and we have  $k_\delta^2 = k^2$ .

Our three  $G$ -invariant functions on  $\mathcal{O}$  are given by

$$\eta_i = 2^{i+1} k^2 (s^{2i} + r^{2i} + t^{2i}), \quad i = 1, 2, 3.$$

Let us use the abbreviations  $\rho_i = \partial\rho/\partial\eta_i$  and  $\rho_u = \partial\rho/\partial u$ , where  $i$  is one of  $1, 2, 3$  and  $u$  ranges in  $\{r, s, t\}$ . Then the derivative  $d\rho = \rho_1 d\eta_1 + \rho_2 d\eta_2 + \rho_3 d\eta_3$  gives rise to the following system of equations

$$(4.8) \quad \begin{pmatrix} \rho_s \\ \rho_r \\ \rho_t \end{pmatrix} = 8k^2 \begin{pmatrix} s & 4s^3 & 12s^5 \\ r & 4r^3 & 12r^5 \\ t & 4t^3 & 12t^5 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}.$$

*Remark 4.5.* The constant  $k^2$  only depends on the Lie algebra  $\mathfrak{g}^{\mathbb{C}}$ . If we consider the highest root decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})_+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus S_+^1 \otimes V_+,$$

we notice that  $\langle E_+, E_+' \rangle = 4 + \dim_{\mathbb{C}} V_+$ . On the other hand  $\langle E_+, E_+' \rangle = k^2 \langle E_+, E_+' \rangle_+ = 4k^2$ , so one may use equation (2.4) to find the value of  $k^2$ .

Type	$A_n, C_n$	$B_{(n-1)/2}, D_{n/2}$	$G_2$	$F_4$	$E_6$	$E_7$	$E_8$
$k^2$	$\frac{n+1}{2}$	$\frac{n-2}{2}$	2	$\frac{9}{2}$	6	9	15

**TABLE 5.** The constant  $k^2$ .

4.2.1. *The Regular Orbit of  $\mathfrak{sl}(2, \mathbb{C})^3$ .* The possible choices for  $\rho$  are narrowed by imposing the endomorphism  $J$  in (4.4) to be an almost complex structure. For the moment we restrict our attention to the subalgebra  $\mathfrak{sl}(2, \mathbb{C})^3$ , and in fact we are considering the regular orbit  $\mathcal{O}_{reg} = \mathcal{O}_+ \times \mathcal{O}_0 \times \mathcal{O}_-$  of  $\mathfrak{sl}(2, \mathbb{C})^3$ , where  $\mathcal{O}_\delta$  is the nonzero nilpotent orbit in  $\mathfrak{sl}(2, \mathbb{C})_\delta$ . By Remark 4.3,  $J$  is indeed an endomorphism of  $\mathfrak{sl}(2, \mathbb{C})^3$ . Let  $A \in \mathfrak{sl}(2, \mathbb{C})^3$  and put  $\xi_A^\delta = \llbracket AX_\delta \rrbracket$ . In Lemma 4.1 we expressed the Kähler form  $\omega_I$  in terms of the functions  $\eta_i$ ,  $i = 1, 2, 3$ . We now use the functions  $\eta_\delta$  defined by

$$\eta_\delta = \eta_1(X_\delta) = k^2 \langle X_\delta, X_\delta' \rangle_{\mathfrak{sl}},$$

where  $\mathfrak{sl}$  is short for  $\mathfrak{sl}(2, \mathbb{C})$ . Notice that  $\eta_+ = 4k^2 s^2$  etc.

The Kähler form can be written as

$$\begin{aligned} \omega_I(\xi_A, \xi_B)_X = 2k^2 \operatorname{Im}\{ & \rho_+ \langle \xi_A^+, \sigma \xi_B^+ \rangle_{\mathfrak{sl}} + \rho_0 \langle \xi_A^0, \sigma \xi_B^0 \rangle_{\mathfrak{sl}} + \rho_- \langle \xi_A^-, \sigma \xi_B^- \rangle_{\mathfrak{sl}} \\ & + \rho_{++} k^2 \langle \xi_A^0, X_+' \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^+, X_+ \rangle_{\mathfrak{sl}} \\ & + \rho_{+0} k^2 (\langle \xi_A^+, X_+' \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^0, X_0 \rangle_{\mathfrak{sl}} + \langle \xi_A^0, X_0' \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^+, X_+ \rangle_{\mathfrak{sl}}) \\ & + \rho_{+-} k^2 (\langle \xi_A^+, X_+' \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^-, X_- \rangle_{\mathfrak{sl}} + \langle \xi_A^-, X_-' \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^+, X_+ \rangle_{\mathfrak{sl}}) \\ & + \rho_{00} k^2 \langle \xi_A^0, X_0' \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^0, X_0 \rangle_{\mathfrak{sl}} \\ & + \rho_{0-} k^2 (\langle \xi_A^0, X_0' \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^-, X_- \rangle_{\mathfrak{sl}} + \langle \xi_A^-, X_-' \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^0, X_0 \rangle_{\mathfrak{sl}}) \\ & + \rho_{--} k^2 \langle \xi_A^-, \sigma X_- \rangle_{\mathfrak{sl}} \langle \sigma \xi_B^-, X_- \rangle_{\mathfrak{sl}} \}, \end{aligned}$$

where  $\rho_{++} = \partial \rho_+ / \partial \eta_+$ , etc. The endomorphism  $J$  is given by

$$\begin{aligned} J_X(\xi_A^+) = -2\rho_+ \llbracket X_+ \sigma \xi_A^+ \rrbracket - 2k^2 \langle \sigma \xi_A^+, X_+ \rangle_{\mathfrak{sl}} (\rho_{++} \llbracket X_+ X_+' \rrbracket \\ + \rho_{+0} \llbracket X_0 X_0' \rrbracket + \rho_{+-} \llbracket X_- X_-' \rrbracket) \end{aligned}$$

and similarly for  $\xi_A^0, \xi_A^-$ . Evaluating at  $H_+$  and  $E_+$  we find that  $J_X H_+ = -4s\rho_+ E_+$  and

$$J_X E_+ = 2s(\rho_+ + \eta_+ \rho_{++}) H_+ + 2\eta_+ s^{-1} (r^2 \rho_{+0} H_0 + t^2 \rho_{+-} H_-).$$

If  $J^2$  is to be  $-1$ , then the  $\mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_-$ -component of the following must vanish,

$$(4.9) \quad J_X^2 H_+ = -8s^2 \rho_+ (\rho_+ + \eta_+ \rho_{++}) H_+ - 8\rho_+ \eta_+ (r^2 \rho_{+0} H_0 + t^2 \rho_{+-} H_-).$$

Thus  $0 = 2\rho_+ \rho_{+0} = \partial(\rho_+^2) / \partial \eta_0$ , so  $\rho_{+0} = 0$ , and similarly  $\rho_{+-} = 0$ . We conclude that  $J^2 = -1$  on  $\mathfrak{sl}(2, \mathbb{C})^3$  if and only if  $\rho_{+0} = \rho_{+-} = \rho_{0-} = 0$ , in which case each of the  $\mathfrak{sl}(2, \mathbb{C})$ -summands is preserved.

Consequently, the condition  $J^2 = -1$  is equivalent to

$$k^2 = 2\eta_\delta \rho_\delta (\rho_\delta + \eta_\delta \rho_{\delta\delta}) = \frac{\partial}{\partial \eta_\delta} (\eta_\delta \rho_\delta)^2$$

for each value of  $\delta$ . Hence,

$$(4.10) \quad \rho_\delta^2 = \left( k^2 \eta_\delta + \frac{c_\delta}{4} \right) / \eta_\delta^2,$$

for some constant  $c_\delta$ . We demand that  $c_\delta \geq 0$  in order for  $\rho_\delta$  to be defined for all  $X_\delta$ . Notice that in terms of the parameter  $s$ , we get

$$(4.11) \quad \rho_s^2 = 16k^4 + c_+/s^2,$$

and similarly for  $\rho_r$  and  $\rho_t$ .

*Remark 4.6.* If we had considered the case  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$  with inner product  $k^2 \langle \cdot, \cdot \rangle$  and used the methods described in Section 4.1 with one invariant function,  $\eta = \eta_1$ , then the potential of the nonzero nilpotent orbit in  $\mathfrak{sl}(2, \mathbb{C})$  would have been given by equation (4.10) and the corresponding metric would have looked like

$$(4.12) \quad \frac{g(\xi_A, \xi_B)_X}{\sqrt{4k^2\eta + c}} = \frac{k^2}{\eta^2} \operatorname{Re} \left( \eta \langle \xi_A, \xi_B' \rangle - \frac{2k^2\eta + c}{4k^2\eta + c} \langle \xi_A, X' \rangle \langle \xi_B', X \rangle \right).$$

Moreover, the potential is a hyperKähler potential if and only if  $c_\delta = 0$  (see [15]).

**Proposition 4.7.** *An  $SU(2)^3$ -invariant hyperKähler structure of the principal orbit of  $\mathfrak{sl}(2, \mathbb{C})^3$  that admits a Kähler potential and has the Kirillov-Kostant-Souriau complex symplectic form, is a product of  $SU(2)$ -invariant structures (given in (4.12)) on each factor.  $\square$*

**4.2.2. Other Lie Algebras.** Recall that we are considering a nilpotent orbit  $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$  such that  $X \in \mathcal{O}$  is decomposed via equation (4.7). In §4.2.1 we described the action of  $J$  on tangent vectors  $\xi_A \in \mathfrak{sl}(2, \mathbb{C})^3$  and found that the potential must satisfy equation (4.11) for  $J$  to be almost complex. To get a complete picture we need to consider the case when  $\xi_A$  lies in the Killing-orthogonal complement of  $\mathfrak{sl}(2, \mathbb{C})^3$ . Additional obstructions in choosing  $\rho$  may emerge this way. As discussed in Section 2.3  $X_\delta$  is a nilpotent element of  $\mathfrak{g}^{\mathbb{C}}$  and via (2.2) lie in the minimal orbit of  $\mathfrak{g}^{\mathbb{C}}$ . By Proposition 2.8 we have a Killing-orthogonal decomposition corresponding to a highest root,

$$(4.13) \quad \mathfrak{g}^{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})_\delta \oplus \mathfrak{k}_\delta^{\mathbb{C}} \oplus (S_\delta^1 \otimes V_\delta).$$

Recall that  $V_\delta$  is a non-trivial  $\mathfrak{k}_\delta^{\mathbb{C}}$ -module and that  $\mathfrak{k}_\delta^{\mathbb{C}}$  is the centralizer of  $\mathfrak{sl}(2, \mathbb{C})_\delta$ . Note that  $\mathfrak{sl}(2, \mathbb{C})_- \subset \mathfrak{k}_+^{\mathbb{C}}$ , so, as an  $\mathfrak{sl}(2, \mathbb{C})_-$ -module,  $V_+$  may consist of trivial and fundamental  $\mathfrak{sl}(2, \mathbb{C})_-$ -modules; similarly for  $\mathfrak{sl}(2, \mathbb{C})_0$ . As we will see below, it suffices to consider the following two situations:

(i) Assume the existence of a trivial  $\mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_-$ -module  $\mathbb{C}^r$  in  $V_+$ . Choose the dimension  $r$  to be maximal, so that the real structure  $\sigma$  preserves the module  $S_+^1 \otimes \mathbb{C}^r$ . We now describe the action of  $\sigma$ . Using the identification  $S_+^1 \otimes \mathbb{C}^r = S_1 \oplus \dots \oplus S_r$ ,  $S_j \cong S_+^1$ , let  $\sigma_{ik}$  be the component of  $\sigma$  that sends  $S_i$  to  $S_k$ . Note that  $j \circ \sigma$  ( $j \in \mathbb{H} \cong S_+^1$ ) commutes with the  $\mathfrak{sl}(2, \mathbb{C})$ -action and by Schur's Lemma is a scalar, hence  $\sigma_{ik} = \mu_{ik} j$ ,  $\mu_{ik} \in \mathbb{C}$ . Since  $\sigma$  is an involution,  $(\mu_{ik})(\overline{\mu_{ik}})^t = -1$  and  $\mathbb{C}^r$  admits a quaternionic structure,  $j: e_i \mapsto \sum_{k=1}^r \mu_{ik} e_k$ . Thus  $\sigma$  acts on  $S_+^1 \otimes \mathbb{C}^r$  as  $j \otimes j$ .

To determine  $J$  on the module  $S_+^1 \otimes \mathbb{C}^r$  we choose a basis on which  $E_+$  acts as  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . In this picture any tangent vector  $\xi_A = \llbracket AX \rrbracket \in S_+^1 \otimes \mathbb{C}^r$  has the form  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes w$ . We now see the convenience of the positions of ' $\xi_A$ ' in (4.5). We immediately get

$$J_X \xi_A = -2s(\rho_1 + 4s^2\rho_2 + 12s^4\rho_3) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes jw = -4k^2\rho_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes jw.$$

Hence, as an endomorphism of  $S_+^1 \otimes W$ ,  $J^2$  is  $-1$  if and only if  $\rho_s^2 = 16k^4$ . Comparing with equation (4.11) we now know that  $c_+ = 0$  if  $V_+$  has a trivial  $\mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_-$ -module.

(ii) Next, consider the case of a tangent vector  $\xi_A$  lying in a  $\mathfrak{sl}(2, \mathbb{C})^3$ -module  $S_+^1 \otimes S_-^1$ . This is a submodule of  $\mathfrak{g}^{\mathbb{C}}$  contained in the Killing orthogonal complement of  $\mathfrak{sl}(2, \mathbb{C})^3$ . It is convenient to choose a basis such that  $E_{\pm}$  acts on  $S_{\pm}^1$  as  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $\sigma = j \otimes j$ . In other words,  $X$  acts as

$$(4.14) \quad s \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \text{Id} + t \text{Id} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We choose two independent vectors that span the image of  $\text{ad}_X$

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi_2 = s \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now if  $A \in S_+^1 \otimes S_-^1$  then  $\xi_A|_X = \llbracket AX \rrbracket$  is in the span of  $\xi_1$  and  $\xi_2$  and using (4.14) the action of  $X$  and  $X'$  on these vectors is easily found to be

$$\llbracket X\xi_1 \rrbracket = 0, \quad \llbracket X'\xi_1 \rrbracket = \xi_2', \quad \llbracket X\xi_2 \rrbracket = 2st\xi_1, \quad \llbracket X'\xi_2 \rrbracket = -(s^2 + t^2)\xi_1'.$$

Using (4.5) we immediately get

$$\begin{aligned} J_X \xi_1 &= -2(\rho_1 + 4\rho_2(s^2 + t^2) + 12\rho_3(s^4 + s^2t^2 + t^4))\xi_2 \\ J_X \xi_2 &= 2(\rho_1(s^2 + t^2) + 4\rho_2(s^4 + t^4) + 12\rho_3(s^6 + t^6))\xi_1, \end{aligned}$$

which via (4.8) may be expressed as

$$J_X \xi_1 = -\frac{s\rho_s - t\rho_t}{4k^2(s^2 - t^2)}\xi_2 \quad \text{and} \quad J_X \xi_2 = \frac{s\rho_s + t\rho_t}{4k^2}\xi_1.$$

Hence,

$$J^2 \xi_1 = -\frac{s^2\rho_s^2 - t^2\rho_t^2}{16k^4(s^2 - t^2)}\xi_1$$

and via (4.11) we see that  $J^2$  is  $-1$  on the module  $S_+^1 \otimes S_-^1$  if and only if  $c_+ = c_-$ .

With these two considerations at hand we are ready to draw the conclusions. By Table 4 there are four orbits to consider:  $(2^3, 1^{n-6})$  in  $\mathfrak{sl}(n, \mathbb{C})$ ,  $(2^6, 1^{n-12})$  in  $\mathfrak{so}(n, \mathbb{C})$ ,  $(2^3, 1^{2n-6})$  in  $\mathfrak{sp}(n, \mathbb{C})$  and  $200000$  in  $\mathfrak{e}_7^{\mathbb{C}}$ . We thus need to decompose  $V_+$  under the action of  $\mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_-$  when  $G$  is a classical Lie group or the exceptional Lie Group  $E_7$ . These



representations can be found in [15], and are listed in Table 6 for convenience. Notice that in the case  $G = SO(n)$  the centralizer is  $\mathfrak{k}_+^c = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n-4, \mathbb{C})$ , and a priori there are two choices of  $\mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_- \subset \mathfrak{k}_+^c$ . One consists in taking  $\mathfrak{sl}(2, \mathbb{C})_0 = \mathfrak{sl}(2, \mathbb{C})$  and  $\mathfrak{sl}(2, \mathbb{C})_- \subset \mathfrak{so}(n-4, \mathbb{C})$  and the other  $\mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_- \subset \mathfrak{so}(n-4, \mathbb{C})$ . However, the first one is not an option because it would allow the appearance of copy of  $S_+^1 \otimes S_0^1 \otimes S_-^1$  and Jordan normal form would therefore be incorrect. We conclude that  $\mathfrak{g}^c$  always contains

$G$	$\mathfrak{k}_+^c$	$V_+$
$Sp(n)$ ( $n \geq 3$ )	$\mathfrak{sp}(n-1, \mathbb{C})$	$S_0^1 \oplus S_-^1 \oplus \mathbb{C}^{2n-6}$
$SU(n)$ ( $n \geq 6$ )	$\mathbb{C} \oplus \mathfrak{sl}(n-2, \mathbb{C})$	$S_0^1 \oplus S_-^1 \oplus \mathbb{C}^{n-6}$
$SO(n)$ ( $n \geq 12$ )	$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n-4, \mathbb{C})$	$\mathbb{C}^2 \otimes (S_0^1 \oplus S_-^1 \oplus \mathbb{R}^{n-12})$
$E_7$	$\mathfrak{so}(12, \mathbb{C})$	$\mathbb{C}^8 \otimes (S_0^1 \oplus S_-^1)$

**TABLE 6.** The centralizer  $\mathfrak{k}_+^c$  for classical groups and  $E_7$ . The module  $V_+$  is decomposed under the action of  $\mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_-$ .

copies of  $S_+^1 \otimes S_0^1$  and  $S_+^1 \otimes S_-^1$ , that copies of  $S_+^1 \otimes S_0^1 \otimes S_-^1$  never occur, and that  $V_+$  contains a trivial  $\mathfrak{sl}(2, \mathbb{C})_0 \oplus \mathfrak{sl}(2, \mathbb{C})_-$ -module unless  $\mathfrak{g}$  is one of  $\mathfrak{sp}(3)$ ,  $\mathfrak{su}(6)$ ,  $\mathfrak{so}(12)$  or  $\mathfrak{e}_7$ . In particular the constants  $c_s$  of the potential coincide,

$$c := c_+ = c_0 = c_-,$$

and if  $V_+$  does not have a trivial summand we get a one-parameter family of hyperKähler metrics with Kähler potential. Alternatively the constant  $c$  must be zero and we get a unique potential.

**4.2.3. An Explicit Potential.** If  $c_+ = c_0 = c_- = 0$  the potential is  $\rho = 4k^2(s+t+r)$  up to addition of some constant. This can be expressed in terms of  $\eta_1$ ,  $\eta_2$  and  $\eta_3$ . To make the notation less cumbersome we introduce the functions

$$\tilde{\eta}_i = \frac{\eta_i}{2^{i+1}k^2} = r^{2i} + s^{2i} + t^{2i}, \quad i = 1, 2, 3.$$

Moreover we put

$$\begin{aligned} \alpha &= \sqrt{\tilde{\eta}_1^3 - 3\tilde{\eta}_1\tilde{\eta}_2 + 2\tilde{\eta}_3}, \quad \beta = 9\alpha^2 - 5\tilde{\eta}_1^3 + 9\tilde{\eta}_1\tilde{\eta}_2, \\ \psi &= \beta^2 + 2(\tilde{\eta}_1^2 - 3\tilde{\eta}_2)^3 \quad \text{and} \\ \kappa &= 2^{\frac{1}{3}} \left( \beta + \sqrt{\psi} \right)^{\frac{1}{3}} + 2\tilde{\eta}_1 - \frac{2^{\frac{2}{3}}(\tilde{\eta}_1^2 - 3\tilde{\eta}_2)}{(\beta + \sqrt{\psi})^{\frac{1}{3}}}. \end{aligned}$$

The potential is now given by

$$(4.15) \quad \rho = \frac{2\sqrt{2}k^2}{\sqrt{3}} \left( \sqrt{\kappa} + \sqrt{\frac{12\alpha}{\sqrt{\kappa}} + 6\tilde{\eta}_1 - \kappa} \right).$$

This was obtained by noticing that  $\lambda = \frac{\rho}{4k^2} = r + s + t$  is a solution of the equation

$$(\lambda^2 - \tilde{\eta}_1)^2 - \frac{8\alpha}{\sqrt{6}}\lambda - 2(\tilde{\eta}_1^2 - \tilde{\eta}_2) = 0.$$

**4.2.4. Relation with Hermitian Symmetric Spaces.** Recall that we labelled the nilpotent orbits of cohomogeneity three according to whether or not the Lie span  $\{X, X'\}$  lies in a  $\sigma$ -invariant  $\mathfrak{sl}(2, \mathbb{C})^3$ -subalgebra for an arbitrary element  $X$ . It will be shown in § 4.3 that the orbit  $(3, 2^2)$  in  $\mathfrak{so}(7, \mathbb{C})$  does not meet the requirement to be a generic cohomogeneity-three orbit. As for the remaining orbits in Table 4 we use an interesting relation with Hermitian symmetric spaces to confirm the generic cohomogeneity-three property.

*Remark 4.8.* It is only the orbit in  $E_7$  that calls for these considerations. In fact, the discussion in § 3.3 covers any cohomogeneity-three nilpotent orbit in a classical Lie algebra different from  $\mathfrak{so}(7, \mathbb{C})$ .

Let  $\mathfrak{r} \subset \mathfrak{g}^{\mathbb{C}}$  be a real semisimple Lie algebra with Cartan decomposition  $\mathfrak{r} = \mathfrak{k} + \mathfrak{p}$  and corresponding Cartan involution  $\theta$ . That is  $\mathfrak{k} = \mathfrak{r}^{\theta}$ ,  $\mathfrak{p} = \mathfrak{r}^{-\theta}$  are such that

$$(4.16) \quad [\mathfrak{k}\mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}\mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}\mathfrak{p}] \subseteq \mathfrak{k}$$

and  $\mathfrak{k} + i\mathfrak{p}$  is a compact real form of  $\mathfrak{r}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$ . In particular we choose  $\mathfrak{r}$  such that  $\mathfrak{k} + i\mathfrak{p} = \mathfrak{g}$  so that  $\sigma$  becomes the conjugation map of  $\mathfrak{r}^{\mathbb{C}}$  with respect to  $\mathfrak{k} + i\mathfrak{p}$ . Let  $R \subset G^{\mathbb{C}}$  be the connected Lie group with Lie algebra  $\mathfrak{r}$ . Then  $K$  is a Lie subgroup of  $R$  with Lie algebra  $\mathfrak{k}$  and, provided the center of  $R$  is finite, the homogeneous space  $R/K$  is a Riemannian symmetric space (of non-compact type).

**Definition 4.9.** A pair  $(\mathfrak{r}, H_0)$  consisting of a real semisimple Lie algebra  $\mathfrak{r}$  with Cartan decomposition  $\mathfrak{r} = \mathfrak{k} + \mathfrak{p}$  and an element  $H_0$  in the center  $Z(\mathfrak{k})$  of  $\mathfrak{k}$  such that the restriction of  $\text{ad}_{H_0}$  to  $\mathfrak{p}$  is a complex structure, is called a *semisimple Lie algebra of Hermitian type*.

Notice, if such an element  $H_0$  exists then  $J = \text{ad}_{H_0}$  clearly induces a  $R$ -invariant almost complex structure on  $R/K$ . On the other hand, if  $R/K$  is Hermitian it follows that  $\mathfrak{r}$  is of Hermitian type (see eg. [12, 18]).

For the rest of this section we assume  $(\mathfrak{r}, H_0)$  to be of Hermitian type; consequently  $\mathfrak{g}^{\mathbb{C}}$  is no longer random among the complex simple

Lie algebras. With the assignment  $J_0 = \text{ad}_{H_0}|_{\mathfrak{p}}$  the complexified Lie algebra decomposes as  $\mathfrak{r}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+ + \mathfrak{p}^-$ , where  $\mathfrak{p}^{\pm}$  is the  $\pm i$  eigenspace of  $J_0$  and  $\mathfrak{p}^- = \overline{\mathfrak{p}^+}$ . From (4.16) it follows that

$$[\mathfrak{k}^{\mathbb{C}} \mathfrak{p}^{\pm}] = \mathfrak{p}^{\pm}, \quad [\mathfrak{p}^+ \mathfrak{p}^-] \subseteq \mathfrak{k}^{\mathbb{C}}, \quad [\mathfrak{p}^{\pm} \mathfrak{p}^{\pm}] = 0.$$

So  $\mathfrak{p}^{\pm}$  are in fact abelian subalgebras of  $\mathfrak{r}^{\mathbb{C}}$ . Let  $\mathfrak{h}$  be a maximal abelian subalgebra of  $\mathfrak{k}$ , and notice that  $H_0 \in Z(\mathfrak{k}) \subseteq \mathfrak{h}$ . Since  $J_0 Z \neq 0$  for nonzero  $Z \in \mathfrak{p}$ , the centralizer of  $H_0$  in  $\mathfrak{r}$  equals  $\mathfrak{k}$ . Thus  $\mathfrak{h}$  is a maximal abelian subalgebra of  $\mathfrak{r}$  and, because the elements of  $\mathfrak{h}^{\mathbb{C}}$  are semisimple,  $\mathfrak{h}^{\mathbb{C}}$  is a Cartan subalgebra of  $\mathfrak{r}^{\mathbb{C}}$ . Let  $\Phi$  denote the root system of  $\mathfrak{r}^{\mathbb{C}}$  relative to  $\mathfrak{h}^{\mathbb{C}}$ . Let moreover  $\Phi^c$  stand for the set of all compact roots, and  $\Phi^{\pm}$  denote positive/negative non-compact roots, respectively. In other words  $\Phi = \Phi^c \cup \Phi^+ \cup \Phi^-$  with

$$\alpha(H_0) = \begin{cases} 0 & \alpha \in \Phi^c \\ \pm i & \alpha \in \Phi^{\pm} \end{cases}.$$

We then have  $\mathfrak{p}^{\pm} = \sum_{\alpha \in \Phi^{\pm}} \mathfrak{r}_{\alpha}^{\mathbb{C}}$  where  $\mathfrak{r}_{\alpha}^{\mathbb{C}}$  is the root space of  $\mathfrak{r}^{\mathbb{C}}$  corresponding to  $\alpha \in \Phi$ , and one finds that  $\overline{Z}$  lies in  $\mathfrak{r}_{-\alpha}^{\mathbb{C}}$  whenever  $Z$  lies in  $\mathfrak{r}_{\alpha}^{\mathbb{C}}$ . Using that  $\mathfrak{p}^+$  is abelian it is easy to show that (see [12])

**Lemma 4.10.** *Let  $\mathfrak{r}$  be a semisimple Lie algebra of hermitian type with Cartan decomposition  $\mathfrak{r} = \mathfrak{k} + \mathfrak{p}$ , and let  $\mathfrak{h} \subseteq \mathfrak{k}$  be a Cartan subalgebra of  $\mathfrak{r}$ . Then there exist  $r \in \mathbb{N}$  and  $r$  linearly independent positive non-compact roots  $\beta_1, \dots, \beta_r$  relative to  $\mathfrak{h}^{\mathbb{C}}$  such that  $\beta_j \pm \beta_k$  is not a root for any  $j \neq k$  and  $\mathfrak{a} = \sum_{j=1}^r \mathbb{C}(E_j + \overline{E}_j)$  is a maximal abelian subalgebra of  $\mathfrak{p}$  ( $E_j$  is a nonzero  $\beta_j$ -root vector).  $\square$*

*Remark 4.11.* The dimension of a maximal abelian subalgebra of  $\mathfrak{p}$  is called the *real rank* of  $\mathfrak{r}$ .

In the above Lemma we may in fact choose  $E_j$  such that  $[E_j E_j \overline{E}_j] = -2E_j$ . Seeing that  $E_j' = -\overline{E}_j$  we obtain a  $\sigma$ -invariant subalgebra isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$  via the complex span of  $\{E_j, \overline{E}_j, [E_j \overline{E}_j]\}$ . Thus  $\mathfrak{a}^{\mathbb{C}}$  is a  $\sigma$ -invariant subalgebra consisting of  $r$  copies of  $\mathfrak{sl}(2, \mathbb{C})$  where  $r$  is the real rank of  $\mathfrak{r}$ . It is a fact that any two maximal abelian subalgebras of  $\mathfrak{p}$  are Ad-conjugate by an element  $k \in K$ . Hence

$$(4.17) \quad \mathfrak{p} = \bigcup_{k \in K} \text{Ad}_k(\mathfrak{a})$$

and by the action of  $K$  any element of  $\mathfrak{p}^+$  can be moved into a  $\sigma$ -invariant  $\mathfrak{sl}(2, \mathbb{C})^r$ -subalgebra.

It turns out that the restricted root system of  $\mathfrak{r}$  (relative to  $\mathfrak{a}$ ) is of type  $BC_r$  or  $C_r$  (see eg. [18, p. 110]). The only system that arises in

our context is  $C_r$  (see Table 7); it is given by

$$(4.18) \quad \{\pm\xi_j \pm \xi_k, \pm 2\xi_j \mid 1 \leq j, k \leq r, j \neq k\},$$

where  $\{\xi_j \mid 1 \leq j \leq r\}$  is the dual basis of  $\{X_j = E_j + \overline{E}_j \mid 1 \leq j \leq r\}$ . Consider the  $j$ 'th  $\mathfrak{sl}(2, \mathbb{C})$  subalgebra of  $\mathfrak{a}^{\mathbb{C}}$  and decompose  $\mathfrak{g}^{\mathbb{C}}$  under the action of this subalgebra. If this is a highest root decomposition,

$$\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})_j \oplus \mathfrak{k}_j^{\mathbb{C}} \oplus (S_j^1 \otimes V_j).$$

we see via (4.18) that

$$(4.19) \quad V_j \cong \mathbb{C}^{\nu} \otimes \left( \bigoplus_{k \neq j} S_k^1 \right),$$

where  $\nu \geq 1$  is the dimension of the root space corresponding to the restricted root  $\pm\xi_j \pm \xi_k$ .

$\mathfrak{r} \subset \mathfrak{g}^{\mathbb{C}}$	$\mathfrak{k}$
$\mathfrak{su}(3, 3) \subset \mathfrak{sl}(6, \mathbb{C})$	$\mathbb{R} \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3)$
$\mathfrak{sp}(3, \mathbb{R}) \subset \mathfrak{sp}(3, \mathbb{C})$	$\mathbb{R} \oplus \mathfrak{su}(3)$
$\mathfrak{so}^*(12) \subset \mathfrak{so}(12, \mathbb{C})$	$\mathbb{R} \oplus \mathfrak{su}(6)$
$\mathfrak{e}_{7(-25)} \subset \mathfrak{e}_7^{\mathbb{C}}$	$\mathbb{R} \oplus \mathfrak{e}_6$

**TABLE 7.** Selected simple Lie algebras of Hermitian type of real rank 3.

We are now in position to verify Table 4. First, let  $\mathfrak{r}$  be one of the Hermitian symmetric Lie algebras of Table 7 and keep in mind the Beauville-data in Tables 2 & 3 of the nilpotent orbit  $\mathcal{O}$  of cohomogeneity three in the corresponding complexified Lie algebra. It is then evident that the  $K$ -module  $\mathfrak{n}$  equals  $\mathfrak{p}^+$  (or  $\mathfrak{p}^-$ ). Since any element  $X$  of  $\mathcal{O}$  can be moved into  $\mathfrak{n}$  by the action of  $G$  we conclude via (4.17) that  $X$  can be put into a  $\mathfrak{sl}(2, \mathbb{C})^3$ -subalgebra. Thus  $\mathcal{O}$  is a generic cohomogeneity three orbit. In addition, since  $X$  may be decomposed as in (4.7), the isomorphism (4.19) gives another argument for the existence of the one-parameter family of hyperKähler metrics with Kähler potentials. Second, let  $\mathfrak{g}$  be one of the Lie algebras  $\mathfrak{sp}(3+i)$ ,  $\mathfrak{su}(6+i)$ ,  $\mathfrak{so}(12+i)$  with  $i \geq 1$ . A look at Tables 2 & 3 reveals that, in spite of the growth of  $K$  as  $i$  increases, the action of  $K$  on  $\mathfrak{n}$  is independent of  $i \geq 0$ . In other words, moving an element of  $\mathfrak{n}$  via the action of  $K$  reduces to the situation where  $i = 0$ . But this is already accounted for by the appearance of a Lie algebra of Hermitian type. This covers all the orbits in Table 4.

**4.3. The Special Cohomogeneity Three Orbit.** Recall that  $\mathfrak{so}(7, \mathbb{C})$  possesses one nilpotent orbit  $\mathcal{O} = (3, 2^2)$  of cohomogeneity three. Unfortunately this orbit behaves differently from the ones of § 4.2. In fact for a typical element  $X \in \mathcal{O}$ ,  $X$  and  $X'$  span the whole ambient Lie algebra  $\mathfrak{so}(7, \mathbb{C})$  (see [20]). Thus there is no alternative left than computing the endomorphism  $J$  on all of  $\mathfrak{so}(7, \mathbb{C})$ . For this task we use MATHEMATICA to derive equations (4.21), (4.22a) and (4.22b) below. The complete computations are collected in a notebook file [20]. In principle, these equations are computable by hand as well, using (4.5) and (4.20).

Via the action of the compact group  $G = SO(7)$  it is possible to move any element  $X$  of  $\mathcal{O}$  into the subspace  $\mathfrak{n}$ . However, this can be improved. The weighted Dynkin diagram of  $\mathcal{O}$  is  $10 \Rightarrow 1$  so there is a basis  $\{\alpha, \beta, \gamma\}$  of simple positive roots such that  $\text{ad}_H$  acts on  $\mathfrak{g}_\alpha^{\mathbb{C}}$ ,  $\mathfrak{g}_\beta^{\mathbb{C}}$  and  $\mathfrak{g}_\gamma^{\mathbb{C}}$  with eigenvalues 1, 0 and 1 respectively. The eigenspaces of  $\text{ad}_H$  with eigenvalues larger than one are  $\mathfrak{g}^{\mathbb{C}}(2) = \mathfrak{g}_{\alpha+\beta+\gamma}^{\mathbb{C}} \oplus \mathfrak{g}_{\beta+2\gamma}^{\mathbb{C}}$  and  $\mathfrak{g}^{\mathbb{C}}(3) = \mathfrak{g}_{\alpha+\beta+2\gamma}^{\mathbb{C}} \oplus \mathfrak{g}_{\alpha+2\beta+2\gamma}^{\mathbb{C}}$ . The zero-eigenspace is  $\mathfrak{g}^{\mathbb{C}}(0) = \mathbb{C}_+ \oplus \mathbb{C}_- \oplus \mathfrak{sl}(2, \mathbb{C})$  so  $K = U(1)_+ U(1)_- SU(1)$  and one finds that  $\mathfrak{g}^{\mathbb{C}}(2) = L_+ + L_-^2$  and  $\mathfrak{g}^{\mathbb{C}}(3) = L_+ L_- S^1 \mathbb{C}^2$  as a  $K$ -module. Thus, fixing nonzero root vectors  $E_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}$  etc., the action of  $K \subset G$  allows us to write a generic element as  $X = r E_{\alpha+\beta+\gamma} + s E_{\beta+2\gamma} + t E_{\alpha+\beta+2\gamma}$  with  $r, s, t > 0$ .

We choose to describe  $\mathfrak{so}(7, \mathbb{C})$  as the set of complex  $(n \times n)$  matrices  $Z$  such that  $ZB + ZB^t = 0$  where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_3 \\ 0 & I_3 & 0 \end{pmatrix}.$$

The diagonal elements form the Cartan subalgebra and the compact real form is  $\sigma : Z \mapsto -\overline{Z}^t$ . For each positive root  $\phi$  the root vector  $E_\phi$  is chosen such that  $\{H_\phi, E_\phi, F_\phi\}$  is a standard basis of  $\mathfrak{sl}(2, \mathbb{C})$  with  $H_\phi = \llbracket E_\phi F_\phi \rrbracket$ ,  $F_\phi = -E_\phi'$ . Explicitly,  $X$  takes the matrix form

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{2}r & 0 & 0 \\ -\sqrt{2}r & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & -t & 0 & s \\ 0 & 0 & 0 & 0 & 0 & -s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The three invariant functions evaluated at  $X$  are  
(4.20)

$$\begin{aligned} \eta_1(X) &= 10 (2r^2 + s^2 + t^2), \quad \eta_2(X) = 20 (2r^4 + 4r^2 t^2 + (s^2 + t^2)^2) \\ \text{and } \eta_3(X) &= 40 (2r^6 + 9r^4 t^2 + 6r^2 t^2 (s^2 + t^2) + (s^2 + t^2)^3). \end{aligned}$$

Our aim is to use (4.5) to compute the endomorphism  $J^2+1$  on a basis of the tangent space at  $X$  and then impose  $J$  to be almost complex. The complex dimension of  $\mathcal{O}$  is 12 therefore the condition  $J^2+1=0$  implies 144 new equations. A minority of these are nontrivial, and in particular one finds that the Kähler potential  $\rho = \rho(r, s, t)$  must solve the following system of equations,

(4.21)

$$\rho_s \rho_{rs} + \rho_t \rho_{rt} = \rho_s \rho_{st} + \rho_t \rho_{tt} = \rho_r \rho_{rs} + 4 \rho_t \rho_{st} = \rho_r \rho_{rt} + 4 \rho_t \rho_{tt} = 0$$

and

$$(4.22a) \quad t(100 - \rho_t^2) = (r \rho_r + s \rho_s) \rho_t$$

$$(4.22b) \quad t(r \rho_r - s \rho_s) = (4r^2 - s^2) \rho_t.$$

Consequently,  $\rho_s^2 + \rho_t^2$  is a function of  $s$  and  $\rho_r^2 + 4\rho_t^2$  is a function of  $r$ . A combination of (4.22a) and (4.22b) gives an equation involving quadratic terms of the first derivatives

$$(4.23) \quad s^2(\rho_s^2 + \rho_t^2 - 100) = r^2(\rho_r^2 + 4\rho_t^2 - 400).$$

Notice that the left hand side of (4.23) is a function of  $s$ , whereas the right is a function of  $r$ , so both must be constant. Thus

$$(4.24a) \quad \rho_s^2 + \rho_t^2 = 100 \left(1 + \frac{c}{s^2}\right),$$

$$(4.24b) \quad \rho_r^2 + 4\rho_t^2 = 100 \left(4 + \frac{c}{r^2}\right)$$

for some constant  $c$ .

Eliminating  $\rho_r$  and  $\rho_s$  from the equations (4.22a), (4.22b) and (4.24a), we obtain a quadratic equation in  $\rho_t^2$ ,

$$50t^4 = t^2 \left( (4r^2 + s^2 + t^2) + 2c \right) \rho_t^2 - \frac{(4r^2 + s^2 + t^2)^2 - 16r^2s^2}{200} \rho_t^4.$$

In order for  $\rho_t^2$  to be real for all  $r, s, t$  one needs that  $c \geq 0$ . Solving for  $\rho_t$  leads to

(4.25)

$$\frac{50t^2}{\rho_t^2} = \varepsilon \sqrt{4r^2s^2 + c(4r^2 + s^2 + t^2) + c^2} + c + \frac{(4r^2 + s^2 + t^2)}{2}$$

where  $\chi = \pm 1$  and  $\varepsilon = \pm 1$ . Integrating, we find that  $\mathcal{O}$  carries a one-parameter family of hyperKähler metrics with Kähler potentials,

(4.26)

$$\begin{aligned} \rho(r, s, t; c) = & 10\chi \left( \sqrt{4r^2 + s^2 + t^2 + 2h(r, s, t)} + f(r, s) \right. \\ & \left. - \sqrt{c} \log \left( h(r, s, t) + \sqrt{c} \sqrt{4r^2 + s^2 + t^2 + 2h(r, s, t)} \right) \right) \end{aligned}$$

where

$$h(r, s, t) = c + \varepsilon \sqrt{4r^2 s^2 + (4r^2 + s^2 + t^2) c + c^2}$$

and  $f(r, s)$  is a function to be determined. The choice of the positive solution ( $\chi = 1$ ) forces one to take  $\varepsilon = 1$ , since  $\rho$  should be everywhere differentiable.

To find  $f$  one may write the right hand side of (4.26) as  $\mathcal{I}(r, s, t) + f(r, s)$  so that  $f_s = \lim_{t \rightarrow 0} (\rho_s - \mathcal{I}_s)$ . The derivative  $\rho_s$  can be found using the equations (4.24a) and (4.25). It turns out that

$$f(r, s) = \sqrt{c} \log(r s)$$

up to addition of some constant.

For  $c = 0$  the solution simplifies somewhat,

$$(4.27) \quad \rho(r, s, t; 0) = 10 \sqrt{(2r + s)^2 + t^2}.$$

Let us introduce a new invariant function on  $\mathcal{O}$ ,

$$\zeta_3(Z) = \frac{1}{5} \eta_1(Z^2) = \frac{1}{5} \langle Z^2, \sigma Z^2 \rangle = \text{Tr } Z Z \overline{Z^t Z^t}, \quad Z \in \mathcal{O}$$

which is non-trivial since  $\mathcal{O}$  consists of 3-step nilpotent elements. Evaluated at  $X$  we get that  $\zeta_3(X) = 4r^4$ . It is convenient to define  $\zeta_j(Z) = \eta_j(Z)/5$ ,  $j = 1, 2$ ; notice that  $\zeta_1(Z) = \text{Tr } Z \overline{Z^t}$  and  $\zeta_2(Z) = \text{Tr } [Z \overline{Z^t}]^2$ . Now (4.27) can be written as

$$(4.28) \quad \rho = 5 \sqrt{2} \sqrt{\zeta_1 + 2\sqrt{\zeta_3} + 2\sqrt{\zeta_1^2 - \zeta_2 - 2\zeta_3}},$$

This is the unique hyperKähler potential, because for  $\lambda \in \mathbb{C}$  it is obvious that  $\zeta_1(\lambda Z) = |\lambda|^2 \zeta_1(Z)$  and  $\zeta_2(\lambda Z) = |\lambda|^4 \zeta_2(Z)$ ,  $i = 2, 3$  so  $\rho(\lambda Z) = |\lambda| \rho(Z)$  (see [5]).

In fact it is possible to write down the one-parameter family of Kähler potentials in terms of the globally defined  $G$ -invariant functions  $\zeta_1$ ,  $\zeta_2$  and  $\zeta_3$ ,

$$(4.29) \quad \rho = \frac{10}{\sqrt{2}} \left( \sqrt{\zeta_1 + 2\sqrt{\zeta_3} + 4h} - \sqrt{2} c \log \left( \frac{h + \frac{c}{\sqrt{2}} \sqrt{\zeta_1 + 2\sqrt{\zeta_3} + 4h}}{\sqrt{\zeta_1^2 - \zeta_2 - 2\zeta_3}} \right) \right),$$

where now

$$h = c + \sqrt{\frac{1}{4} (\zeta_1^2 - \zeta_2 - 2\zeta_3) + c \left( \frac{1}{2} \zeta_1 + \sqrt{\zeta_3} \right) + c^2}.$$

*Remark 4.12.* Using hyperKähler quotients Kobak & Swann [16] found the invariant hyperKähler potential for the hyperKähler structure on

the nilpotent orbit  $(\mathcal{O}_{(3,2^2)}, \Sigma) \subset \mathfrak{so}(n, \mathbb{C})$ . The expression given in [16] matches what we found in (4.28).

*Remark 4.13.* Kobak & Swann showed in [13] the existence of a one-to-one correspondence between the nilpotent orbits  $\mathcal{O}_{(2^4)} \subset \mathfrak{so}(8, \mathbb{C})$  and  $\mathcal{O}_{(3,2^2)} \subset \mathfrak{so}(7, \mathbb{C})$ . The nilpotent orbit  $\mathcal{O}_{(2^4)} \subset \mathfrak{so}(8, \mathbb{C})$  is of cohomogeneity two and admits a one-parameter family of  $SO(8)$ -invariant hyperKähler metrics with  $SO(8)$ -invariant Kähler potentials which includes the unique hyperKähler potential (see [15]). Reducing the symmetry group to  $SO(7)$  the above shows that there are no extra solutions, even though the cohomogeneity changes to three.

**4.4. The Main Theorem.** In § 4.2 and § 4.3 we considered hyperKähler metrics with Kähler potential on all nilpotent orbits of cohomogeneity three under the action of the compact group  $G$ . We proved that the majority of orbits allowed only one such metric. Exceptions were found either when the Beauville bundle is a vector bundle over one of the following (compact) Hermitian symmetric spaces

$$\frac{SU(6)}{S(U_3 \times U_3)}, \quad \frac{Sp(3)}{U(3)}, \quad \frac{SO(12)}{U(6)}, \quad \frac{E_7}{E_6 U(1)}$$

or in the special case  $G = SO(7)$ . To summarize,

**Theorem 4.14.** *Suppose  $G$  is a compact simple Lie group and  $\mathcal{O}$  is a nilpotent orbit in  $\mathfrak{g}^{\mathbb{C}}$  of cohomogeneity three. Then  $\mathcal{O}$ , endowed with the Kirillov-Kostant-Souriau complex symplectic form  $\Sigma$ , admits a unique  $G$ -invariant hyperKähler metric with  $G$ -invariant hyperKähler potential. In fact, the metric is the unique  $G$ -invariant hyperKähler metric of  $(\mathcal{O}, \Sigma)$  with a Kähler potential unless  $\mathfrak{g}$  is one of  $\mathfrak{sp}(3)$ ,  $\mathfrak{su}(6)$ ,  $\mathfrak{so}(12)$ ,  $\mathfrak{e}_7$ ,  $\mathfrak{so}(7)$ , in which case the metric lies in a one-parameter family of hyperKähler metrics with Kähler potential.*

*The unique hyperKähler potential is given by (4.15) for the generic cohomogeneity-three orbits, and by (4.29) for the special orbit.*

*Proof.* The only thing left to consider is the issue of the  $G$ -invariant hyperKähler potential. First, consider the generic cohomogeneity-three orbits. By Swann [19] it is known that any hyperKähler metric on  $\mathcal{O}$  admits a hyperKähler potential. Consequently, when the  $G$ -invariant Kähler potential  $\rho$  is unique ( $c = 0$ ) there is nothing to prove. Consider now the cases where  $\mathcal{O}$  admits a one-parameter family of  $G$ -invariant Kähler potentials. Now any  $X \in \mathcal{O}$  lies in a real  $\mathfrak{sl}(2, \mathbb{C})^3$  subalgebra, so if  $\rho$  is a hyperKähler potential it must restrict to a hyperKähler potential of the regular orbit  $\mathcal{O}_{reg}$  of  $\mathfrak{sl}(2, \mathbb{C})^3$ . But the hyperKähler metric on  $\mathcal{O}_{reg}$  is a product of three hyperKähler metrics on the minimal



orbit of  $\mathfrak{sl}(2, \mathbb{C})$ , and on each factor the hyperKähler potential is unique ( $c = 0$ ), see Remark 4.6. At last, for the special cohomogeneity three orbit we showed that the hyperKähler potential is given by (4.28).  $\square$

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